

High-order expansions of the Detweiler-Whiting singular field in Schwarzschild spacetime

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The self-field of a charged particle has a singular component that diverges at the particle. We use both coordinate and covariant approaches to compute an expansion of this singular field for particles in generic geodesic orbits about a Schwarzschild black hole for scalar, electromagnetic and gravitational cases. We check both approaches yield identical results and give, as an application, the calculation of previously unknown mode-sum regularisation parameters.

In the so-called *mode-sum regularization* approach to self-force calculations, each mode of the retarded field is finite, while their sum diverges. The sum may be rendered finite and convergent by the subtraction of appropriate *regularization parameters*. Higher order parameters lead to faster convergence in the mode-sum. To demonstrate the significant benefit they yield, we use our newly derived parameters to calculate a highly accurate value of $F_r = 0.000013784482575667959(3)$ for the self-force on a scalar particle in a circular orbit around a Schwarzschild black hole.

Finally, as a second example application of our high-order expansions, we compute high-order expressions for use in the effective source approach to self-force calculations.

I. INTRODUCTION

The notion of a self-force has a long history in physics. Early work led to the derivation of the Abraham-Lorentz-Dirac [1] formula for the radiation reaction force on an accelerating point electric charge moving in a flat spacetime. In the 1960s, DeWitt and Brehme [2] derived the curved spacetime equivalent, and a correction was later provided by Hobbs [3]. For a comprehensive review of the self-force problem, see Refs. [4–6].

With the advent of gravitational wave astronomy, the past two decades have seen a surge in interest in the self-force problem. This has been motivated by the study of so-called Extreme Mass Ratio Inspiral (EMRI) systems. These binary systems – consisting of a compact object of ~ 1 solar mass spiralling into a massive black hole of $\sim 10^6$ solar masses – are one of the most promising candidates for study by planned space-based gravitational wave detectors [7–13]. The nature of interferometric gravitational wave detectors is that unlike traditional electromagnetic spectrum telescopes they are not directional; instead they see all directions at once. As a result, a crucial component of any observation is the use of data analysis techniques such as matched filtering to disentangle the desired signal from the noise. This, in turn, requires an accurate model of the signal one expects to see. This is even more true given the recent proposals for the eLISA/NGO detector [14] (which will have a reduced sensitivity compared to LISA, the previously proposed space-based gravitational wave detector) will mean that accurate waveform models are all the more crucial.

The past two decades have seen new derivations of the equations of motion of a point charge moving in a curved spacetime; this was first done in the gravitational charge case by Mino, Sasaki and Tanaka [15] and Quinn and Wald [16] and in the minimally coupled scalar charge case by Quinn [17]. Since then, there have been significant achievements in putting these derivations on a firmer footing. The use of distributional sources in Einsteins equations is known to be problematic [18]. Nevertheless, with sufficient care they have proven to be a useful practical tool in many cases [19, 20]. In the case of the self-force problem recent derivations have avoided the introduction of distributional sources altogether through the use of matched asymptotic expansions and careful limiting procedures [21–26]. At first perturbative order it is satisfying that these more rigorous derivations reproduce the same equations of motion as one would obtain using distributional sources. Nevertheless, it is likely that these more rigorous methods are required to advance to second perturbative order [27–31] and beyond [32–34].

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Several practical self-force computation strategies have developed from these formal derivations, all of which are based on the now-justified assumption that the use of a distributional source is acceptable at first perturbative order:

- The *mode-sum* approach [35, 36]
- The *effective source* approach [37, 38]
- The *matched expansion* approach [39, 40]

The key to all three approaches is the subtraction of an appropriate *singular* component (which only appears due to the use of a distributional source) from the retarded field to leave a finite *regular* field which is solely responsible for the self-force. This singular component must satisfy two criteria:

1. It has the same singular structure as the full retarded field on the particle's worldline.
2. It does not contribute to the self-force.

There are many choices for a singular field that satisfies the above criteria, although not all choices are equal. Detweiler and Whiting [41] identified a particularly appropriate choice. Through a Green function decomposition, they derived a singular field which not only satisfies the above two criteria, but also has the property that when it is subtracted from the full retarded field, it leaves a regularized field which is a solution to the *homogeneous* wave equation. Extensions of this idea to second perturbative order [29, 31] and fully non-perturbative contexts [34] have recently been developed.

In this paper, we develop highly accurate approximations to the Detweiler-Whiting singular field of point scalar and electromagnetic charges as well as that of a point mass. This is achieved through high-order series expansions in a parameter ϵ , which acts as a measure of distance from the particle's world-line. As examples, we derive explicit expressions for the case of geodesic motion in Schwarzschild spacetime and show how these may be applied to both the mode-sum and effective source approaches described above. Similar expansions may also be used for the quasi-local component in the matched expansion approach [42–44] whereby the Green function is matched onto a quasi-normal mode sum and the retarded field is then computed as the integral of this matched retarded Green function along the worldline of the particle.

While the primary focus of this paper is on computing the singular field for Schwarzschild spacetime, many of the expressions we give are valid in more general spacetimes. In particular, where space allows, we do not make any assumptions about the spacetime being Ricci-flat. To make this distinction explicit, we use the Weyl tensor, C_{abcd} , in expressions which are valid only in vacuum and the Riemann tensor, R_{abcd} in expressions which are also valid for non-vacuum spacetimes. Note that this is done only for space reasons; our raw calculations include all non-vacuum terms in addition to those given in this paper and we have made the full expressions available in electronic form [45].

The layout of this paper is as follows. In Sec. II we give exact formal expressions for the singular field in scalar, electromagnetic and gravitational cases. In Sec. III we describe covariant and coordinate approaches to the computation of expansions of the key fundamental bitensors appearing in the formal expressions for the singular field. In Sec. IV we give explicit expressions for the singular field in the form of covariant series expansions. Our equivalent coordinate expressions are too large to be of use in print; instead we have made them available electronically [45]. In Sec. V, we use our coordinate expansions to derive high-order regularization parameters for use in the mode-sum method. In doing so, we give the next two previously unknown non-zero regularization parameters in scalar, electromagnetic and gravitational cases. In Sec. VI we apply our coordinate expansions to the effective source method and compute an effective source for our high order singular field. This improves on previous effective source calculations by adding an additional four orders, bringing the approximation to eight-from-leading order. In Sec. VII we summarize our results and discuss further prospects for their application. Finally, in Appendices A and B we give explicit expressions for many of the expansions in coordinate and covariant form, respectively, along with a more in-depth derivation of our expressions.

Throughout this paper, we use units in which $G = c = 1$ and adopt the sign conventions of [46]. We denote symmetrization of indices using parenthesis (e.g. (ab)), anti-symmetrization using square brackets (e.g. $[ab]$), and exclude indices from (anti-)symmetrization by surrounding them by vertical bars (e.g. $(a|b|c)$, $[a|b|c]$). We denote pairwise (anti-)symmetrization using an overbar, e.g. $\overline{R}_{(ab\overline{cd})} = \frac{1}{2}(R_{abcd} + R_{cdab})$. Capital letters are used to denote the spinorial/tensorial indices appropriate to the field being considered.

In many of our calculations, we have several spacetime points to be considered. Our convention is that (see Fig. 1):

- the point x refers to the point where the field is evaluated
- the point x' refers to an arbitrary spacetime point
- the point \bar{x} refers to an arbitrary point on the world-line

- the point $x_{(\text{adv})}$ refers to the advanced point of x on the world-line
- the point $x_{(\text{ret})}$ refers to the retarded point of x on the world-line

In computing expansions, we use ϵ as an expansion parameter to denote the fundamental scale of separation, so that $x - x' \approx x - \bar{x} \approx x' - x = \mathcal{O}(\epsilon)$. Where tensors are to be evaluated at one of these points, we decorate their indices appropriately using ' and $\bar{}$, e.g. T^a , $T^{a'}$ and $T^{\bar{a}}$ refer to tensors at x , x' and \bar{x} , respectively.

II. SINGULAR FIELD AND SELF-FORCE OF A POINT PARTICLE

In an appropriate gauge, the retarded field, $\varphi^A(x)$, of an arbitrary point particle satisfies the inhomogeneous wave equation with a distributional source,

$$\mathcal{D}^A_B \varphi^B = -4\pi Q \int u^A \delta_4(x, z(\tau')) d\tau', \quad (2.1)$$

where

$$\mathcal{D}^A_B = \delta^A_B (\square - m^2) - P^A_B \quad (2.2)$$

is the wave operator, $\square \equiv g^{ab} \nabla_a \nabla_b$, g^{ab} is the (contravariant) metric tensor, ∇_a is the covariant derivative defined by a connection \mathcal{A}^A_{Ba} : $\nabla_a \varphi^A = \partial_a \varphi^A + \mathcal{A}^A_{Ba} \varphi^B$, m is the mass of the field, Q is the charge of the particle, $P^A_B(x)$ is a potential term and $u^A(x)$ is an appropriate tensor constructed from products of the four-velocity on the world-line parallel transported to x . The retarded solutions to this equation give rise to a field which exerts a self-force

$$F^a = p^a_A \varphi_{(\text{ret})}^A \quad (2.3)$$

on the particle, where $p^a_A(x)$ is a tensor at x and depends on the type of charge. The distributional nature of the source leads to this self-force being divergent at the location of the particle and a regularisation scheme must be employed. We require a *singular* field, $\varphi_{(\text{S})}^A$, that captures the singular behaviour of φ^A and that, when subtracted from φ^A , leaves a finite *physical* self-force.

Detweiler and Whiting [41] showed how such a singular field can be constructed through a Green function decomposition. In four spacetime dimensions and within a normal neighborhood, the Green function for the retarded/advanced solutions to Eq. (2.1) may be given in Hadamard form,

$$G_{(\text{ret})/(\text{adv})}^A B' (x, x') = \theta_{-/+}(x, x') \{ U^A B' (x, x') \delta[\sigma(x, x')] - V^A B' (x, x') \theta[-\sigma(x, x')] \}, \quad (2.4)$$

where $\delta[\sigma(x, x')]$ is the covariant form of the Dirac delta function, $\theta[\sigma(x, x')]$ is the Heaviside step function, while $U^A B' (x, x')$ and $V^A B' (x, x')$ are symmetric bi-spinors/tensors which are regular for $x' \rightarrow x$. The bi-scalar $\sigma(x, x')$ is the Synge [4] world function; it is equal to one half of the squared geodesic distance between x and x' . The first term here, involving $U^A B' (x, x')$, represents the *direct* part of the Green function while the second term, involving $V^A B' (x, x')$, is known as the *tail* part of the Green function.

Detweiler and Whiting proposed to define a *singular* Green function by taking the symmetric Green function, $G_{(\text{sym})}^A B' = \frac{1}{2}(G_{(\text{ret})}^A B' + G_{(\text{adv})}^A B')$ and adding $V^A B' (x, x')$ (a homogeneous solution to Eq. (2.1)). This leads to the singular Green function,

$$G_{(\text{S})}^A B' (x, x') = \frac{1}{2} \{ U^A B' (x, x') \delta[\sigma(x, x')] + V^A B' (x, x') \theta[\sigma(x, x')] \}. \quad (2.5)$$

Note that this has support on and *outside* the past and future light-cone (i.e. for points x and x' spatially separated) and is only uniquely defined provided x and x' are within a convex normal neighbourhood. Given this singular Green function, we may define the Detweiler-Whiting singular field,

$$\varphi_{(\text{S})}^A = \int_{\tau_{(\text{adv})}}^{\tau_{(\text{ret})}} G_{(\text{S})}^A B' (x, z(\tau')) u^{B'} d\tau', \quad (2.6)$$

which also satisfies Eq. (2.1). Subtracting this singular field from the retarded field, we obtain the *regularized* field,

$$\varphi_{(\text{R})}^A = \varphi_{(\text{ret})}^A - \varphi_{(\text{S})}^A, \quad (2.7)$$

which Detweiler and Whiting showed gives the correct finite physical self-force,

$$F^a = p^a{}_A \varphi_{(R)}^A. \quad (2.8)$$

Moreover, this regularized field is a solution of the homogeneous wave equation,

$$\mathcal{D}^A{}_B \varphi_{(R)}^B = 0. \quad (2.9)$$

This holds independently of whether one is considering a scalar or electromagnetically charged point particle or a point mass. To make this more explicit, in the following subsections we give the form these expressions take in each of scalar, electromagnetic and gravitational cases.

A. Scalar Case

In the scalar case the singular field, $\Phi^{(S)}$, is a solution of the inhomogeneous scalar wave equation,

$$(\square - \xi R - m^2) \Phi^{(S)} = q \int \sqrt{-g} \delta_4(x, z(\tau)) d\tau, \quad (2.10)$$

where q is the scalar charge and ξ is the coupling to the background scalar curvature. An expression for $\Phi^{(S)}$ may be found by considering the scalar Green function [obtained by taking $g^A{}_{B'} = 1$ in Eq. (2.5)],

$$G^{(S)} = \frac{1}{2} \{ U(x, x') \delta[\sigma(x, x')] + V(x, x') \theta[\sigma(x, x')] \}, \quad (2.11)$$

which is a solution of the equation

$$(\square - \xi R - m^2) G^{(S)} = -4\pi \delta(x, x'). \quad (2.12)$$

Given this expression for the Green function, the scalar singular field is

$$\begin{aligned} \Phi^{(S)}(x) &= q \int_{\gamma} G^{(S)}(x, z(\tau)) d\tau \\ &= \frac{q}{2} \left[\frac{U(x, x')}{\sigma_{c'} u^{c'}} \right]_{x'=\bar{x}_{(ret)}}^{x'=x_{(adv)}} + \frac{q}{2} \int_{\tau_{(ret)}}^{\tau_{(adv)}} V(x, z(\tau)) d\tau \end{aligned} \quad (2.13)$$

and one computes the scalar self-force from the regularized scalar field $\Phi^{(R)} = \Phi^{(ret)} - \Phi^{(S)}$ as

$$F^a = g^{ab} \Phi_{,b}^{(R)}. \quad (2.14)$$

B. Electromagnetic Case

In Lorenz gauge, the electromagnetic singular field satisfies the equation

$$\square A_a^{(S)} - R_a{}^b A_b^{(S)} = -4\pi e \int g_{aa'} u^{a'} \sqrt{-g} \delta_4(x, z(\tau)) d\tau, \quad (2.15)$$

where e is the electric charge. An expression for $A_a^{(S)}$ may be found by considering the electromagnetic Green function [obtained by taking $g^A{}_{B'} = g^a{}_{a'}$ in Eq. (2.5)],

$$G_{aa'}^{(S)}(x, x') = \frac{1}{2} \{ U(x, x')_{aa'} \delta(\sigma(x, x')) + V(x, x')_{aa'} \theta(\sigma(x, x')) \}, \quad (2.16)$$

where $U^a{}_{a'} = \Delta^{1/2} g^a{}_{a'}$. This Green function is a solution of the equation

$$\square G_{aa'}^{(S)} - R_a{}^b G_{ba'}^{(S)} = -4\pi g_{aa'} \delta_4(x, x'). \quad (2.17)$$

Given this expression for the Green function, the electromagnetic singular field is

$$\begin{aligned} A_a^{(S)} &= e \int_{\gamma} G_{aa'}^{(S)}(x, z(\tau')) u^{a'} d\tau' \\ &= \frac{e}{2} \left[\frac{u^{a'} U_{aa'}(x, x')}{\sigma_{c'} u^{c'}} \right]_{x'=x_{(\text{ret})}}^{x'=x_{(\text{adv})}} + \frac{e}{2} \int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V_{aa'}(x, z(\tau)) u^{a'} d\tau. \end{aligned} \quad (2.18)$$

One computes the electromagnetic self-force from the electromagnetic regular field, $A_a^{(R)} = A_a^{(\text{ret})} - A_a^{(S)}$, as

$$F^a = g^{ab} u^c A_{[c,b]}^{(R)}. \quad (2.19)$$

C. Gravitational Case

In Lorenz gauge, the trace-reversed singular first order metric perturbation satisfies the equation

$$\square \bar{h}_{ab}^{(S)} + 2C_a{}^c{}_b{}^d \bar{h}_{cd}^{(S)} = -16\pi\mu \int g_{a'(a} u^{a'} g_{b)b'} u^{b'} \sqrt{-g} \delta_4(x, z(\tau)) d\tau, \quad (2.20)$$

where μ is the mass of the particle and the trace-reversed singular field is related to the non-trace-reversed version by $\bar{h}_{ab}^{(S)} = h_{ab}^{(S)} - \frac{1}{2} h^{(S)} g_{ab}$ with $h^{(S)} = g^{ab} h_{ab}^{(S)}$. An expression for $\bar{h}_{ab}^{(S)}$ may be found by considering the gravitational Green function [obtained by taking $g^A{}_{B'} = g^{(a}{}_{a'} g^{b)}{}_{b'}$ in Eq. (2.5)],

$$G_{aba'b'}^{(S)}(x, x') = \frac{1}{2} \{ U(x, x')_{aba'b'} \delta[\sigma(x, x')] + V(x, x')_{aba'b'} \theta[\sigma(x, x')] \}, \quad (2.21)$$

with $U^{ab}{}_{a'b'} = \Delta^{1/2} g^{(a}{}_{a'} g^{b)}{}_{b'}$. This Green function is a solution of the equation

$$\square G_{aba'b'}^{(S)} + 2C_a{}^p{}_b{}^q G_{pq a'b'}^{(S)} = -4\pi g_{a'(a} g_{b)b'} \delta_4(x, x'). \quad (2.22)$$

Given this expression for the Green function, the trace-reversed singular first order metric perturbation is

$$\begin{aligned} \bar{h}_{ab}^{(S)} &= 4\mu \int_{\gamma} G_{aba'b'}^{(S)}(x, z(\tau')) u^{a'} u^{b'} d\tau' \\ &= 2\mu \left[\frac{u^{a'} u^{b'} U_{aba'b'}(x, x')}{\sigma_{c'} u^{c'}} \right]_{x'=x_{(\text{ret})}}^{x'=x_{(\text{adv})}} + 2\mu \int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V_{aba'b'}(x, z(\tau)) u^{a'} u^{b'} d\tau. \end{aligned} \quad (2.23)$$

One computes the gravitational self-force from the regularized trace-reversed singular first order metric perturbation, $\bar{h}_{ab}^{(R)} = \bar{h}_{ab}^{(\text{ret})} - \bar{h}_{ab}^{(S)}$, as

$$F^a = k^{abcd} \bar{h}_{bc;d}^{(R)}, \quad (2.24)$$

where

$$k^{abcd} \equiv \frac{1}{2} g^{ad} u^b u^c - g^{ab} u^c u^d - \frac{1}{2} u^a u^b u^c u^d + \frac{1}{4} u^a g^{bc} u^d + \frac{1}{4} g^{ad} g^{bc}. \quad (2.25)$$

III. COVARIANT AND COORDINATE EXPANSION OF FUNDAMENTAL BITENSORS

In the previous section, we gave expressions for the singular field in terms of the bitensors $U^A{}_{B'}(x, x')$ and $V^A{}_{B'}(x, x')$. The first of these is given by

$$U^{AB'}(x, x') = \Delta^{1/2}(x, x') g^{AB'}(x, x'), \quad (3.1)$$

where $\Delta(x, x')$ is the Van Vleck-Morette determinant [4],

$$\Delta(x, x') = -[-g(x)]^{-1/2} \det(-\sigma_{;\alpha\beta'}(x, x')) [-g(x')]^{-1/2} = \det \left(-g^{\alpha'}{}_{\alpha}(x, x') \sigma^{\alpha}{}_{\beta'}(x, x') \right), \quad (3.2)$$

$g^{AB'}$ is the bi-tensor of parallel transport appropriate to the tensorial nature of the field, eg.

$$g^{AB'} = \begin{cases} 1 & \text{(scalar)} \\ g^{ab'} & \text{(electromagnetic)} \\ g^{a'(a} g^{b')b'} & \text{(gravitational)}, \end{cases} \quad (3.3)$$

and where the higher spin fields are taken in Lorentz gauge. Here, $g^{\alpha'}{}_\alpha(x, x')$ is the bi-vector of parallel transport defined by the transport equation

$$\sigma^\alpha g_{ab';\alpha} = 0 = \sigma^{\alpha'} g_{ab';\alpha'}. \quad (3.4)$$

The bitensor $V^{AB'}(x, x')$ may be expressed in terms of a formal expansion in increasing powers of σ [47]:

$$V^{AB'}(x, x') = \sum_{n=0}^{\infty} V_n{}^{AB'}(x, x') \sigma^n(x, x'), \quad (3.5)$$

where the coefficients $V_n{}^{AB'}(x, x')$ satisfy the recursion relations

$$\sigma^{\alpha'} (\Delta^{-1/2} V_n{}^{AB'})_{;\alpha'} + (n+1) \Delta^{-1/2} V_n{}^{AB'} + \frac{1}{2n} \Delta^{-1/2} \mathcal{D}^{B'}{}_{C'} V_{n-1}{}^{AC'} = 0, \quad (3.6a)$$

for $n \in \mathbb{N}$, along with the ‘initial condition’

$$\sigma^{\alpha'} (\Delta^{-1/2} V_0{}^{AB'})_{;\alpha'} + \Delta^{-1/2} V_0{}^{AB'} + \frac{1}{2} \Delta^{-1/2} \mathcal{D}^{B'}{}_{C'} (\Delta^{1/2} g^{AC'}) = 0. \quad (3.6b)$$

Looking at the above equations for $U^{AB'}(x, x')$ and $V_n{}^{AB'}(x, x')$, we see that a key component of the present work involves the computation of several fundamental bitensors, in particular, the world function $\sigma(x, x')$, Van Vleck determinant $\Delta^{1/2}(x, x')$, four-velocity $u^a(x)$, and bivector of parallel transport $g_a{}^b(x, x')$. This may be achieved by expressing them as expansions about some arbitrary point \bar{x} which is close to x and x' . We derive these here using both covariant and coordinate methods, each of which has its own advantages and disadvantages. The covariant expression is more elegant, allowing for compact formulas; however these formulas hide complex terms such as high order derivatives of the Weyl tensor that quickly become extremely time consuming to compute, even using computer tensor algebra packages such as GRTensorII [48] or xAct [49]. The coordinate approach is less elegant but more practical for explicit calculations and it avoids the need to use tensor algebra. Independently of the approach taken, these expansions may be used to compute expansions of $U^{AB'}(x, x')$ and $V_n{}^{AB'}(x, x')$ (by substituting into the above equations), and hence of the singular field. In the case of covariant expansions, for explicit calculations one must further expand the covariant expressions in coordinates, yielding an expression which may be directly compared with those obtained from the coordinate approach. The resulting expressions are long but are explicit functions of the coordinates, enabling them to be transformed directly into, for example, C functions, indeed we give them in such form online [45].

A. Covariant Approach

In this subsection, we briefly discuss our method for obtaining covariant expansions for the biscalars appearing in Eqs. (2.13), (2.18) and (2.23). We eventually seek expansions about a point \bar{x} on the worldline (which we may treat as fixed in the majority of this paper). In doing so, we follow the strategy of Haas and Poisson [4, 50]:

- For the generic biscalar $A(x, z(\tau))$, write it as $A(\tau) \equiv A(x, z(\tau))$.
- Compute the expansion about $\tau = \bar{\tau}$. This takes the form

$$A(\tau) = A(\bar{\tau}) + \dot{A}(\bar{\tau})(\tau - \bar{\tau}) + \frac{1}{2} \ddot{A}(\bar{\tau})(\tau - \bar{\tau})^2 + \dots, \quad (3.7)$$

where $\dot{A}(\bar{\tau}) = A_{;\bar{a}} u^{\bar{a}}$, $\ddot{A}(\bar{\tau}) = A_{;\bar{a}\bar{b}} u^{\bar{a}} u^{\bar{b}}$, \dots .

- Compute the covariant expansions of the coefficients $\dot{A}(\bar{\tau})$, $\ddot{A}(\bar{\tau})$, \dots about $\bar{\tau}$.

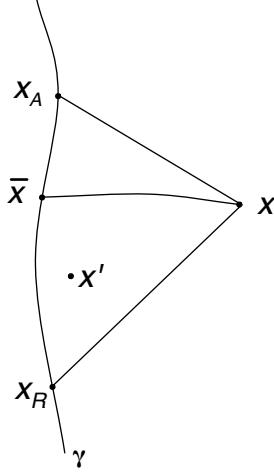


FIG. 1. We expand all bitensors (which are functions of x and x') about the arbitrary point \bar{x} on the worldline.

- Evaluate the expansion at the desired point, eg. $A(x') = A(x, x')$.
- The resulting expansion depends on τ through the powers of $\tau - \bar{\tau}$. Replace these by their expansion in ϵ (about \bar{x}), the distance between x and the world-line.

A key ingredient of this calculation is the expansion of $\Delta\tau \equiv \tau - \bar{\tau}$ in ϵ . The leading orders in this expansion were developed by Haas and Poisson [50] for the particular choices $\Delta\tau_+ \equiv v - \bar{\tau}$ and $\Delta\tau_- \equiv u - \bar{\tau}$. They found

$$\Delta\tau_{\pm} = (\bar{r} \pm \bar{s}) \mp \frac{(\bar{r} \pm \bar{s})^2}{6\bar{s}} R_{u\sigma u\sigma} \mp \frac{(\bar{r} \pm \bar{s})^2}{24\bar{s}} [(\bar{r} \pm \bar{s}) R_{u\sigma u\sigma|u} - R_{u\sigma u\sigma|\sigma}] + \mathcal{O}(\epsilon^5), \quad (3.8)$$

where $\bar{r} \equiv \sigma_{\bar{a}} u^{\bar{a}}$ and $\bar{s} \equiv (g^{\bar{a}\bar{b}} + u^{\bar{a}} u^{\bar{b}}) \sigma_{\bar{a}} \sigma_{\bar{b}}$. In Appendix B we extend their calculation to the higher orders required in the present work. In the same Appendix, we also apply the above method to compute covariant expansions of all quantities appearing in the expression for the singular field.

In order to obtain explicit expressions, we substitute in the coordinate expansion for $\sigma_{\bar{a}}$ (as discussed in Sec. A) along with the metric, Riemann tensor and 4-velocity (all evaluated at \bar{x}). In doing so, we only have to keep terms that contribute up to the required order and truncate any higher order terms.

B. Coordinate Approach

In this subsection we follow a similar approach as described above, but in coordinates. We will start by considering two arbitrary points x and x' near \bar{x} . We will seek expansion where the coefficients are evaluated at \bar{x} , so we introduce the notation

$$\Delta x^a = x^a - x^{\bar{a}}, \quad \delta x^{a'} = x^{a'} - x^a = x^{a'} - \Delta x^a - x^{\bar{a}}, \quad (3.9)$$

where we use the convention that the index carries the information about the point: $\bar{x}^a = x^{\bar{a}}$. In the calculations below Δx^a and $\delta x^{a'}$ are both assumed to be small, of order ϵ .

The first item we require for our calculations is a coordinate expansion of the biscalar $\sigma(x, x')$, the Synge world function. We start with a standard coordinate series expansion about x , see for example [43] (note the difference in convention for Δx^a in that paper), to get

$$\sigma(x, x') = \frac{1}{2} g_{ab}(x) \delta x^{a'} \delta x^{b'} + A_{abc}(x) \delta x^{a'} \delta x^{b'} \delta x^{c'} + B_{abcd}(x) \delta x^{a'} \delta x^{b'} \delta x^{c'} \delta x^{d'} + \dots \quad (3.10)$$

The coefficients are readily determined in terms of derivatives of the metric at x by use of the defining identity $2\sigma = \sigma_{a'} \sigma^{a'}$, see [43]. To be explicit, the first few are given by

$$\begin{aligned} A_{abc}(x) &= \frac{1}{4} g_{(ab,c)}(x) \\ B_{abcd}(x) &= \frac{1}{12} g_{(ab,cd)}(x) - \frac{1}{24} g^{pq}(x) (g_{(ab,|p|}(x) g_{cd),q}(x) - 12 g_{(ab,|p|}(x) g_{|q|c,d)}(x) + 36 g_{p(a,b}(x) g_{|q|c,d)}(x)) . \end{aligned}$$

We now go one step further by expanding the coefficients about \bar{x} to give a double expansion in Δx^a and $\delta x^{a'}$ with coefficients at \bar{x} . The first few terms are

$$\begin{aligned}\sigma(x, x') = & \frac{1}{2}g_{\bar{a}\bar{b}}(\bar{x})\delta x^{a'}\delta x^{b'} + \left[\frac{1}{2}g_{\bar{a}\bar{b},\bar{c}}(\bar{x})\delta x^{a'}\delta x^{b'}\Delta x^c + A_{\bar{a}\bar{b}\bar{c}}(\bar{x})\delta x^{a'}\delta x^{b'}\delta x^{c'} \right] \\ & + \left[\frac{1}{4}g_{\bar{a}\bar{b},\bar{c}\bar{d}}(\bar{x})\delta x^{a'}\delta x^{b'}\Delta x^c\Delta x^d + A_{\bar{a}\bar{b}\bar{c},\bar{d}}(\bar{x})\delta x^{a'}\delta x^{b'}\delta x^{c'}\Delta x^d + B_{\bar{a}\bar{b}\bar{c}\bar{d}}(\bar{x})\delta x^{a'}\delta x^{b'}\delta x^{c'}\delta x^{d'} \right] + O(\epsilon^5),\end{aligned}\quad (3.11)$$

where now we interpret $\delta x^{a'}$ as $x^{a'} - \Delta x^a - x^{\bar{a}}$ and we use square brackets to distinguish terms of different order in ϵ . Rather than disturb the flow here and throughout this section we just give the first few terms of each expansion for a general metric to make the structure clear and give explicit expressions in Schwarzschild space-time to much higher order in Appendix A.

Now that the coefficients are at the fixed point \bar{x} , it is straightforward to take derivatives of σ at x and x' , for example,

$$\begin{aligned}\sigma_{a'} = & g_{\bar{a}\bar{b}}\delta x^{b'} + \left[g_{\bar{a}\bar{b},\bar{c}}\delta x^{b'}\Delta x^c + 3A_{\bar{a}\bar{b}\bar{c}}\delta x^{b'}\delta x^{c'} \right] \\ & + \left[\frac{1}{2}g_{\bar{a}\bar{b},\bar{c}\bar{d}}\delta x^{b'}\Delta x^c\Delta x^d + 3A_{\bar{a}\bar{b}\bar{c},\bar{d}}\delta x^{b'}\delta x^{c'}\Delta x^d + 4B_{\bar{a}\bar{b}\bar{c}\bar{d}}\delta x^{b'}\delta x^{c'}\delta x^{d'} \right] + O(\epsilon^4),\end{aligned}\quad (3.12)$$

$$\sigma_a = -\sigma_{a'} + \frac{1}{2}g_{\bar{b}\bar{c},\bar{a}}\delta x^{b'}\delta x^{c'} + \left[\frac{1}{2}g_{\bar{b}\bar{c},\bar{d}}\delta x^{b'}\delta x^{c'}\delta x^{d'} + A_{\bar{b}\bar{c}\bar{d},\bar{a}}\delta x^{b'}\delta x^{c'}\delta x^{d'} \right] + O(\epsilon^4),\quad (3.13)$$

$$\sigma_{a'b} = -g_{\bar{a}\bar{b}} - g_{\bar{a}\bar{b},\bar{c}}\Delta x^c + \left[(3A_{\bar{a}\bar{c}\bar{d},\bar{b}} - 12B_{\bar{a}\bar{b}\bar{c}\bar{d}})\delta x^{c'}\delta x^{d'} - \frac{1}{2}g_{\bar{a}\bar{b},\bar{c}\bar{d}}\Delta x^c\Delta x^d \right] + O(\epsilon^3).\quad (3.14)$$

Likewise we can calculate the Van Vleck-Morette determinant directly from its definition,

$$\Delta^{\frac{1}{2}}(x, x') = \left(-[-g(x)]^{-\frac{1}{2}} |-\sigma_{a'b}(x, x')| [-g(x')]^{-\frac{1}{2}} \right)^{\frac{1}{2}},\quad (3.15)$$

giving

$$\begin{aligned}\Delta^{\frac{1}{2}}(x, x') = & 1 + \left[\frac{1}{2}g^{\bar{c}\bar{d}}(2g_{\bar{a}\bar{c},\bar{b}\bar{d}} - g_{\bar{a}\bar{b},\bar{c}\bar{d}} - g_{\bar{c}\bar{d},\bar{a}\bar{b}}) + \frac{1}{4}g^{\bar{c}\bar{d}}g^{\bar{e}\bar{f}}(g_{\bar{e}\bar{e},\bar{a}}g_{\bar{d}\bar{f},\bar{b}} + 2g_{\bar{c}\bar{a},\bar{b}}(g_{\bar{e}\bar{f},\bar{d}} - 2g_{\bar{d}\bar{e},\bar{f}}) \right. \\ & \left. + 2g_{\bar{a}\bar{c},\bar{e}}(g_{\bar{b}\bar{d},\bar{f}} - g_{\bar{b}\bar{f},\bar{d}}) - g_{\bar{a}\bar{b},\bar{c}}(g_{\bar{e}\bar{f},\bar{d}} - 2g_{\bar{d}\bar{e},\bar{f}})) \right] \delta x^a\delta x^b + O(\epsilon^3).\end{aligned}\quad (3.16)$$

To obtain an expression at $x_{(\text{adv})}$ and $x_{(\text{ret})}$, we allow $x^{a'}$ to be on the worldline and again give it as an expansion around the point \bar{x} , as shown in Fig. 1. Writing $x^{a'}$ in terms of proper time τ gives

$$x^{a'}(\tau) = x^{\bar{a}} + u^{\bar{a}}\Delta\tau + \frac{1}{2!}\dot{u}^{\bar{a}}\Delta\tau^2 + \frac{1}{3!}\ddot{u}^{\bar{a}}\Delta\tau^3 + \dots,\quad (3.17)$$

where $u^{\bar{a}}$ is the four velocity at the point $x^{\bar{a}}$ and $\Delta\tau = \tau - \bar{\tau}$. We are interested in determining the points on the worldline that are connected to x by a null geodesic, that is we want to solve

$$\begin{aligned}\sigma(x^a, x^{a'}(\tau)) = 0 = & \frac{1}{2}g_{\bar{a}\bar{b}}(u^{\bar{a}}\Delta\tau - \Delta x^a)(u^{\bar{b}}\Delta\tau - \Delta x^b) \\ & + \left[\frac{1}{2}g_{\bar{a}\bar{b}}(u^{\bar{a}}\Delta\tau - \Delta x^a)\dot{u}^{\bar{b}}\Delta\tau^2 + \frac{1}{2}g_{\bar{a}\bar{b},\bar{c}}(u^{\bar{a}}\Delta\tau - \Delta x^a)(u^{\bar{b}}\Delta\tau - \Delta x^b)\Delta x^c \right. \\ & \left. + \frac{1}{4}g_{\bar{a}\bar{b},\bar{c}}(\bar{x})(u^{\bar{a}}\Delta\tau - \Delta x^a)(u^{\bar{b}}\Delta\tau - \Delta x^b)(u^{\bar{c}}\Delta\tau - \Delta x^c) \right] + O(\epsilon^4).\end{aligned}\quad (3.18)$$

By writing $\Delta\tau = \tau_1\epsilon + \tau_2\epsilon^2 + \tau_3\epsilon^3 + \dots$ and explicitly inserting an ϵ in front of Δx^a , we may equate coefficients of powers of ϵ to obtain

$$\tau_1^2 + 2g_{\bar{a}\bar{b}}u^{\bar{a}}\Delta x^b\tau_1 - g_{\bar{a}\bar{b}}\Delta x^a\Delta x^b = 0,\quad (3.19)$$

$$\begin{aligned}g_{\bar{a}\bar{b}}u^{\bar{a}}(u^{\bar{a}}\tau_1 - \Delta x^a)\tau_2 = & - \left[\frac{1}{2}g_{\bar{a}\bar{b}}(u^{\bar{a}}\tau_1 - \Delta x^a)u^{\bar{b}}\tau_1^2 + \frac{1}{2}g_{\bar{a}\bar{b},\bar{c}}(u^{\bar{a}}\tau_1 - \Delta x^a)(u^{\bar{b}}\tau_1 - \Delta x^b)\Delta x^c \right. \\ & \left. + \frac{1}{4}g_{\bar{a}\bar{b},\bar{c}}(\bar{x})(u^{\bar{a}}\tau_1 - \Delta x^a)(u^{\bar{b}}\tau_1 - \Delta x^b)(u^{\bar{c}}\tau_1 - \Delta x^c) \right].\end{aligned}\quad (3.20)$$

Equation (3.19) is a quadratic with two real roots of opposite sign (for x spacelike separated from \bar{x}) corresponding to the first approximation to our points $x_{(\text{adv})}$ and $x_{(\text{ret})}$,

$$\tau_{1\pm} = g_{\bar{a}\bar{b}}u^{\bar{a}}\Delta x^b \pm \sqrt{(g_{\bar{a}\bar{b}}u^{\bar{a}}\Delta x^b)^2 + g_{\bar{a}\bar{b}}\Delta x^a\Delta x^b} \equiv \bar{r}_{(1)} \pm \rho,\quad (3.21)$$

where $\bar{r}_{(1)}$ is the leading order term in the coordinate expansion of the quantity \bar{r} appearing in our covariant expansions. Equation (3.20) is typical of the higher order equations giving τ_n in terms of lower order terms.

IV. EXPANSIONS OF THE SINGULAR FIELD

In this section we list the covariant form of the singular field to order ϵ^4 , where ϵ is the fundamental scale of separation, so, for example, \bar{r} , \bar{s} and $\sigma(x, \bar{x})^{\bar{a}}$ are all of leading order ϵ . The coordinate forms of these expansions are too long to be useful in print form so instead they are available to download [45].

A. Scalar singular field

To $\mathcal{O}(\epsilon^4)$, the scalar singular field is

$$\Phi^{(S)} = q \left\{ \frac{1}{\bar{s}} + \frac{\bar{r}^2 - \bar{s}^2}{6\bar{s}^3} C_{u\sigma u\sigma} + \frac{1}{24\bar{s}^3} \left[(\bar{r}^2 - 3\bar{s}^2)\bar{r} C_{u\sigma u\sigma;u} - (\bar{r}^2 - \bar{s}^2)C_{u\sigma u\sigma;\sigma} \right] + \frac{1}{360\bar{s}^5} [\Phi^{(S)}]_{(3)} + \frac{1}{4320\bar{s}^5} [\Phi^{(S)}]_{(4)} + \mathcal{O}(\epsilon^5) \right\}, \quad (4.1)$$

where

$$\begin{aligned} [\Phi^{(S)}]_{(3)} = & 15 \left[\bar{r}^2 - \bar{s}^2 \right]^2 C_{u\sigma u\sigma} C_{u\sigma u\sigma} + \bar{s}^2 \left[(\bar{r}^2 - \bar{s}^2)(3C_{u\sigma u\sigma;\sigma\sigma} + 4C_{u\sigma\sigma\bar{a}} C_{u\sigma\sigma}^{\bar{a}}) + (\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4)(4C_{u\sigma u\bar{a}} C_{u\sigma u}^{\bar{a}} \right. \\ & \left. + 3C_{u\sigma u\sigma;uu}) + \bar{r}(\bar{r}^2 - 3\bar{s}^2)(16C_{u\sigma u}^{\bar{a}} C_{u\sigma\sigma\bar{a}} - 3C_{u\sigma u\sigma;u\sigma}) \right] + \bar{s}^4 \left[2C_u^{\bar{a}}{}_u{}^{\bar{b}}[(\bar{r}^2 + \bar{s}^2)C_{\sigma\bar{a}\sigma\bar{b}} \right. \\ & \left. + 2\bar{r}(\bar{r}^2 + 3\bar{s}^2)C_{u\bar{a}\sigma\bar{b}}] + 2C_u^{\bar{a}}{}_u{}^{\bar{b}}[2\bar{r}C_{\sigma\bar{a}\sigma\bar{b}} + (\bar{r}^2 + \bar{s}^2)C_{u\bar{b}\sigma\bar{a}}] + [\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4]C_{u\bar{a}u\bar{b}} C_u^{\bar{a}}{}_u{}^{\bar{b}} \right. \\ & \left. + [\bar{r}^2 + \bar{s}^2]2C_{u\bar{a}\sigma\bar{b}} C_u^{\bar{a}}{}_u{}^{\bar{b}} + C_{\sigma\bar{a}\sigma\bar{b}} C_{\sigma}^{\bar{a}}{}_u{}^{\bar{b}} \right] \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} [\Phi^{(S)}]_{(4)} = & 30C_{u\sigma u\sigma} \left[\bar{r}(3\bar{r}^4 - 10\bar{r}^2\bar{s}^2 + 15\bar{s}^4)C_{u\sigma u\sigma;u} - 30(\bar{r}^2 - \bar{s}^2)^2 C_{u\sigma u\sigma;\sigma} \right] + 2\bar{s}^2 \left[3C_{u\sigma u\sigma;uuu} \bar{r}[\bar{r}^4 - 10\bar{r}^2\bar{s}^2 - 15\bar{s}^4] \right. \\ & \left. + 3\bar{r}[\bar{r}^2 - 3\bar{s}^2]C_{u\sigma u\sigma;u\sigma\sigma} - 3[\bar{r}^2 - \bar{s}^2]C_{u\sigma u\sigma;\sigma\sigma\sigma} - 3[\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4]C_{u\sigma u\sigma;uu\sigma} - 9\bar{r}[\bar{r}^4 - 10\bar{r}^2\bar{s}^2 \right. \\ & \left. - 15\bar{s}^4]C_{u\sigma u\bar{a};u} - C_{u\sigma\bar{a}}^{\bar{a}}[18C_{u\sigma\sigma\bar{a};\sigma}(\bar{r}^2 - \bar{s}^2) - \bar{r}(\bar{r}^2 - 3\bar{s}^2)(10C_{u\sigma\sigma\bar{a};u} - 16C_{u\sigma u\bar{a};\sigma} - 5C_{u\sigma u\sigma;\bar{a}}) \right. \\ & \left. - 30C_{u\sigma u\bar{a};u}(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4)] - C_{u\sigma u}^{\bar{a}}[36\bar{r}(\bar{r}^2 - 3\bar{s}^2)C_{u\sigma\sigma\bar{a};\sigma} + (\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4)(13C_{u\sigma u\bar{a};\sigma} + 5C_{u\sigma u\sigma;\bar{a}} \right. \\ & \left. - 25C_{u\sigma\sigma\bar{a};u})] \right] - 12\bar{s}^4 \left[C_{\sigma}^{\bar{a}}{}_u{}^{\bar{b}}[C_{\sigma\bar{a}\sigma\bar{b};\sigma} - \bar{r}(C_{\sigma\bar{a}\sigma\bar{b};u} - 2C_{u\bar{a}\sigma\bar{b};\sigma}) - (\bar{r}^2 + \bar{s}^2)(2C_{u\bar{a}\sigma\bar{b};u} - C_{u\bar{a}u\bar{b};\sigma}) \right. \\ & \left. - \bar{r}(\bar{r}^2 + 3\bar{s}^2)C_{u\bar{a}u\bar{b};u}] + 2C_u^{\bar{a}}{}_u{}^{\bar{b}}[C_{\sigma\bar{a}\sigma\bar{b};\sigma}\bar{r} + (\bar{r}^2 + \bar{s}^2)(C_{u\bar{a}\sigma\bar{b};\sigma} + C_{u\bar{b}\sigma\bar{a};\sigma} - C_{\sigma\bar{a}\sigma\bar{b};u}) \right. \\ & \left. + \bar{r}(\bar{r}^2 + 3\bar{s}^2)(C_{u\bar{a}u\bar{b};\sigma} - C_{u\bar{a}\sigma\bar{b};u} - C_{u\bar{b}\sigma\bar{a};u}) - C_{u\bar{a}u\bar{b}}(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4)] + C_u^{\bar{a}}{}_u{}^{\bar{b}}[(\bar{r}^2 + \bar{s}^2)C_{\sigma\bar{a}\sigma\bar{b};\sigma} \right. \\ & \left. - \bar{r}(\bar{r}^2 + 3\bar{s}^2)(C_{\sigma\bar{a}\sigma\bar{b};u} - 2C_{u\bar{a}\sigma\bar{b};\sigma}) + (\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4)(C_{u\bar{a}u\bar{b};\sigma} - 2C_{u\bar{a}\sigma\bar{b};u}) \right. \\ & \left. - C_{u\bar{a}u\bar{b};u}\bar{r}(\bar{r}^4 + 10\bar{r}^2\bar{s}^2 + 5\bar{s}^4)] \right]. \end{aligned} \quad (4.3)$$

B. Electromagnetic singular field

To $\mathcal{O}(\epsilon^4)$, the electromagnetic singular field is

$$\begin{aligned} A_a^{(S)} = & eg_a^{\bar{a}} \left\{ \frac{u_{\bar{a}}}{\bar{s}} + \frac{1}{6\bar{s}^3} \left[3\bar{r}\bar{s}^2 C_{a\bar{a}u\sigma} + C_{u\sigma u\sigma}(\bar{r}^2 - \bar{s}^2)u_{\bar{a}} \right] + \frac{1}{24\bar{s}^3} \left[4\bar{s}^2(C_{a\bar{a}u\sigma;u} - \bar{r}C_{a\bar{a}u\sigma;\sigma}) \right. \right. \\ & \left. \left. + (\bar{r}(\bar{r}^2 - 3\bar{s}^2)C_{u\sigma u\sigma;u} - (\bar{r}^2 - \bar{s}^2)C_{u\sigma u\sigma\sigma})u_{\bar{a}} \right] + \frac{1}{2880\bar{s}^5} [A_{\bar{a}}^{(S)}]_{(3)} + \frac{1}{25920\bar{s}^5} [A_{\bar{a}}^{(S)}]_{(4)} + \mathcal{O}(\epsilon^5) \right\}, \quad (4.4) \end{aligned}$$

where

$$\begin{aligned} [A_{\bar{a}}^{(S)}]_{(3)} = & 120C_{u\sigma u\bar{a};u\sigma\bar{a}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) - 120C_{u\sigma u\bar{a};\sigma\bar{a}}\bar{r}\bar{s}^4 + 120C_u^{\bar{c}}{}_u{}^{\bar{d}} C_{\sigma\bar{c}\bar{a}\bar{d}}\bar{r}\bar{s}^6 - 240C_{u\sigma u\bar{a}}C_{u\sigma u\sigma}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) \\ & - 360C_{u\sigma\bar{a}\bar{c}}C_{u\sigma u}^{\bar{c}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) - 120C_{u\sigma u\bar{a};uu}\bar{r}\bar{s}^4(\bar{r}^2 + 3\bar{s}^2) + 120\bar{s}^6C_u^{\bar{c}}{}_u{}^{\bar{d}}(C_{u\bar{d}\bar{a}\bar{c}}\bar{r} + C_{\sigma\bar{d}\bar{a}\bar{c}}) \\ & + 40C_{u\bar{c}\bar{a}\bar{d}}C_u^{\bar{c}}{}_u{}^{\bar{d}}\bar{s}^6(3\bar{r}^2 + \bar{s}^2) - 120C_{u\bar{a}\sigma\bar{c}}[C_{u\sigma\sigma}^{\bar{c}}\bar{r}\bar{s}^4 + 2C_{u\sigma u}^{\bar{c}}\bar{s}^4(\bar{r}^2 + \bar{s}^2)] - 120C_{u\bar{a}u\bar{c}}[C_{u\sigma\sigma}^{\bar{c}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) \\ & + C_{u\sigma u}^{\bar{c}}\bar{r}\bar{s}^4(\bar{r}^2 + 3\bar{s}^2)] + \{8C_{\sigma\bar{c}\sigma\bar{d}}C_{\sigma}^{\bar{c}}{}_u{}^{\bar{d}}\bar{s}^4 - 5C_{\bar{c}\bar{d}\bar{e}\bar{f}}C_{\bar{c}\bar{d}\bar{e}\bar{f}}\bar{s}^8 - 24C_{u\sigma u\sigma;u\sigma}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) \\ & + 128C_{u\sigma u}^{\bar{c}}C_{u\sigma\sigma\bar{c}}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) + 24C_{u\sigma u\sigma;\sigma\sigma}\bar{s}^2(\bar{r}^2 - \bar{s}^2) + 32C_{u\sigma\sigma\bar{c}}C_{u\sigma\sigma}^{\bar{c}}\bar{s}^2(\bar{r}^2 - \bar{s}^2) \right. \\ & \left. + 120C_{u\sigma u\sigma}C_{u\sigma u\sigma}(\bar{r}^2 - \bar{s}^2)^2 + 16C_u^{\bar{c}}{}_u{}^{\bar{d}}C_{\sigma\bar{c}\sigma\bar{d}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) + 24C_{u\sigma u\sigma;uu}\bar{s}^2(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4) \right. \\ & \left. + 32C_{u\sigma u\bar{c}}C_{u\sigma u}^{\bar{c}}\bar{s}^2(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4) + 8C_{u\bar{c}\bar{a}\bar{d}}C_u^{\bar{c}}{}_u{}^{\bar{d}}\bar{s}^4(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4) + 16\bar{s}^4C_u^{\bar{c}}{}_u{}^{\bar{d}}[2C_{\sigma\bar{c}\sigma\bar{d}}\bar{r} \right. \\ & \left. + C_{u\bar{d}\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2)] + 16\bar{s}^4C_{u\bar{c}\sigma\bar{d}}[C_u^{\bar{c}}{}_u{}^{\bar{d}}(\bar{r}^2 + \bar{s}^2) + 2C_u^{\bar{c}}{}_u{}^{\bar{d}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)] \right\} u_{\bar{a}} \end{aligned} \quad (4.5)$$

and

$$\begin{aligned}
[A_{\bar{a}}^{(S)}]_{(4)} = & 216C_{u\sigma u\bar{a};\sigma\sigma\bar{a}}\bar{r}\bar{s}^4 + 540C_{u\sigma u\sigma;\sigma}C_{u\sigma u\bar{a}}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) + 720C_{u\sigma u\bar{a};\sigma}C_{u\sigma u\sigma}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) \\
& - 216C_{u\sigma u\bar{a};u\sigma\bar{a}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) + 1512C_{u\sigma\bar{a}}\bar{c};\sigma C_{u\sigma u\bar{c}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) + 216C_{u\sigma u\bar{a};u\sigma\bar{a}}\bar{r}\bar{s}^4(\bar{r}^2 + 3\bar{s}^2) \\
& - 864C_{u\sigma\bar{a}}\bar{c};u C_{u\sigma u\bar{c}}\bar{r}\bar{s}^4(\bar{r}^2 + 3\bar{s}^2) - 216C_{u\sigma u\bar{a};uuu}\bar{s}^4(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4) - 540C_{u\sigma u\sigma;u}C_{u\sigma u\bar{a}}\bar{s}^2(\bar{r}^4 \\
& - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4) + 720C_{u\sigma u\bar{a};u}C_{u\sigma u\sigma}\bar{s}^2(-\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + 3\bar{s}^4) - 432\bar{s}^6C_{\sigma}\bar{c}\bar{d};\sigma(C_{u\bar{c}ud}\bar{r} + C_{u\bar{d}\sigma\bar{c}}) \\
& - 648\bar{s}^6C_{u}\bar{c}\bar{d};\sigma(C_{u\bar{d}\bar{a}\bar{c}}\bar{r} + C_{\sigma\bar{d}\bar{a}\bar{c}}) + 16\bar{s}^8C_{u\bar{a}}\bar{c}\bar{d};\bar{e}(C_{u\bar{c}\bar{d}\bar{e}}\bar{r} - C_{u\bar{e}\bar{c}\bar{d}}\bar{r} + C_{\sigma\bar{c}\bar{d}\bar{e}} - C_{\sigma\bar{e}\bar{c}\bar{d}}) \\
& + 432\bar{s}^4C_{u\sigma\sigma}\bar{c};\sigma[C_{u\bar{a}\sigma\bar{c}}\bar{r} + C_{u\bar{a}\bar{u}\bar{c}}(\bar{r}^2 + \bar{s}^2)] + 504\bar{s}^4C_{u\bar{a}\sigma}\bar{c};\sigma[C_{u\sigma\sigma\bar{c}}\bar{r} + 2C_{u\sigma u\bar{c}}(\bar{r}^2 + \bar{s}^2)] \\
& - 324\bar{s}^4C_{u\sigma\sigma}\bar{c};u[C_{u\bar{a}\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) + C_{u\bar{a}\bar{u}\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)] + 108\bar{s}^4C_{u\sigma u}\bar{c};\sigma[5C_{u\bar{a}\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) + 6C_{u\sigma\bar{a}\bar{c}}(\bar{r}^2 + \bar{s}^2) \\
& + 3C_{u\bar{a}\bar{u}\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)] - 96\bar{s}^4C_{u\bar{a}\sigma}\bar{c};u[3C_{u\sigma\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) + 8C_{u\sigma u\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)] + 96\bar{s}^4C_{u\sigma u}\bar{c};\bar{c}[3C_{u\sigma\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) \\
& + 4C_{u\sigma u\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)] + 24\bar{s}^4C_{u\bar{a}\bar{u}}\bar{c};\sigma[9C_{u\sigma\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) + 17C_{u\sigma u\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)] - 216\bar{s}^4C_{u\sigma u}\bar{c};u[3C_{u\bar{a}\sigma\bar{c}}\bar{r}(\bar{r}^2 \\
& + 3\bar{s}^2) + 6C_{u\sigma\bar{a}\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2) + 2C_{u\bar{a}\bar{u}\bar{c}}\bar{s}^4(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4)] - 72\bar{s}^4C_{u\bar{a}\bar{u}}\bar{c};u[8C_{u\sigma\sigma\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2) \\
& + 7C_{u\sigma u\bar{c}}(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4)] - 216\bar{s}^6C_{u}\bar{c}\bar{a}\bar{d};\sigma[3C_{\sigma\bar{c}\bar{a}\bar{d}}\bar{r} + C_{u\bar{c}\bar{a}\bar{d}}(3\bar{r}^2 + \bar{s}^2)] - 144\bar{s}^6C_{u}\bar{c}\bar{a}\bar{d};\sigma[3C_{u\bar{d}\sigma\bar{c}}\bar{r} \\
& + C_{u\bar{c}ud}(3\bar{r}^2 + \bar{s}^2)] + 48\bar{s}^6C_{u\bar{a}\sigma}\bar{c};\bar{d}[3C_{u\bar{c}\bar{a}\bar{d}}\bar{r} + 3C_{u\bar{d}\sigma\bar{c}}\bar{r} + 3C_{\sigma\bar{c}\sigma\bar{d}} + C_{u\bar{c}ud}(3\bar{r}^2 + \bar{s}^2)] \\
& + 216\bar{s}^6C_{\sigma}\bar{c}\bar{a}\bar{d};u[3C_{u\bar{d}\sigma\bar{c}}\bar{r} + C_{u\bar{c}ud}(3\bar{r}^2 + \bar{s}^2)] + 144\bar{s}^6C_{u}\bar{c}\bar{a}\bar{d};u[3C_{\sigma\bar{d}\bar{a}\bar{c}}\bar{r} + C_{u\bar{d}\bar{a}\bar{c}}(3\bar{r}^2 + \bar{s}^2)] \\
& + 48\bar{s}^6C_{u\bar{a}\bar{u}}\bar{c};\bar{d}[3\bar{r}C_{\sigma\bar{c}\sigma\bar{d}} + 3\bar{r}C_{u\bar{c}ud}(3\bar{r}^2 + \bar{s}^2) + C_{u\bar{c}\sigma\bar{d}}(3\bar{r}^2 + \bar{s}^2) + C_{u\bar{d}\sigma\bar{c}}(3\bar{r}^2 + \bar{s}^2)] \\
& + 216\bar{s}^6C_{u}\bar{c}\bar{a}\bar{d};u[3C_{u\bar{c}ud}\bar{r}(\bar{r}^2 + \bar{s}^2) + C_{u\bar{d}\sigma\bar{c}}(3\bar{r}^2 + \bar{s}^2)] + 144\bar{s}^6C_{u}\bar{c}\bar{a}\bar{d};u[3C_{u\bar{c}\bar{a}\bar{d}}\bar{r}(\bar{r}^2 + \bar{s}^2) + C_{\sigma\bar{c}\bar{a}\bar{d}}(3\bar{r}^2 + \bar{s}^2)] \\
& + \{45C_{\bar{c}\bar{d}\bar{e}\bar{f}}\sigma C_{\bar{c}\bar{d}\bar{e}\bar{f}}\bar{s}^8 - 45C_{\bar{c}\bar{d}\bar{e}\bar{f}}u C_{\bar{c}\bar{d}\bar{e}\bar{f}}\bar{r}\bar{s}^8 + 36C_{u\sigma u\sigma;u\sigma\sigma\bar{r}}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) - 540C_{u\sigma u\sigma;\sigma}C_{u\sigma u\sigma}(\bar{r}^2 - \bar{s}^2)^2 \\
& - 36C_{u\sigma u\sigma;\sigma\sigma\bar{r}}\bar{s}^2(\bar{r}^2 - \bar{s}^2) + 36C_{u\sigma u\sigma;uuu}\bar{r}\bar{s}^2(\bar{r}^4 - 10\bar{r}^2\bar{s}^2 - 15\bar{s}^4) - 36C_{u\sigma u\sigma;uu\sigma}\bar{s}^2(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4) \\
& + 180C_{u\sigma u\sigma;u}C_{u\sigma u\sigma}(3\bar{r}^5 - 10\bar{r}^3\bar{s}^2 + 15\bar{r}\bar{s}^4) - 216\bar{s}^2C_{u\sigma\sigma}\bar{c};\sigma[2\bar{r}C_{u\sigma u\bar{c}}(\bar{r}^2 - 3\bar{s}^2) - 2C_{u\sigma\sigma\bar{c}}(-\bar{r}^2 + \bar{s}^2)] \\
& - 72\bar{s}^4C_{\sigma}\bar{c}\bar{a}\bar{d};\sigma[2C_{u\bar{c}\sigma\bar{d}}\bar{r} + C_{\sigma\bar{c}\sigma\bar{d}} + C_{u\bar{c}ud}(\bar{r}^2 + \bar{s}^2)] - 144\bar{s}^4C_{u}\bar{c}\bar{a}\bar{d};\sigma[C_{\sigma\bar{c}\sigma\bar{d}}\bar{r} + C_{u\bar{c}\sigma\bar{d}}(\bar{r}^2 + \bar{s}^2) \\
& + C_{u\bar{d}\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) + C_{u\bar{c}ud}\bar{r}(\bar{r}^2 + 3\bar{s}^2)] + 72\bar{s}^4C_{\sigma}\bar{c}\bar{a}\bar{d};u[C_{\sigma\bar{c}\sigma\bar{d}}\bar{r} + 2C_{u\bar{c}\sigma\bar{d}}(\bar{r}^2 + \bar{s}^2) + C_{u\bar{c}ud}\bar{r}(\bar{r}^2 + 3\bar{s}^2)] \\
& + 60\bar{s}^2C_{u\sigma\sigma}\bar{c};u[2C_{u\sigma\sigma\bar{c}}\bar{r}(\bar{r}^2 - 3\bar{s}^2) + 5C_{u\sigma u\bar{c}}(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4)] + 72\bar{s}^2C_{u\sigma u}\bar{c};u[3C_{u\sigma u\bar{c}}\bar{r}(\bar{r}^4 - 10\bar{r}^2\bar{s}^2 \\
& - 15\bar{s}^4) + 5C_{u\sigma\sigma\bar{c}}(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4)] - 72\bar{s}^4C_{u}\bar{c}\bar{a}\bar{d};u[C_{\sigma\bar{c}\sigma\bar{d}}(\bar{r}^2 + \bar{s}^2) + 2C_{u\bar{c}\sigma\bar{d}}\bar{r}(\bar{r}^2 + 3\bar{s}^2) \\
& + C_{u\bar{c}ud}(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4)] + 144\bar{s}^4C_{u}\bar{c}\bar{a}\bar{d};u[C_{\sigma\bar{c}\sigma\bar{d}}(\bar{r}^2 + \bar{s}^2) + C_{u\bar{c}\sigma\bar{d}}\bar{r}(\bar{r}^2 + 3\bar{s}^2) + C_{u\bar{d}\sigma\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2) \\
& + C_{u\bar{c}ud}(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4)] - 60\bar{s}^2C_{u\sigma u\sigma}\bar{c};\bar{c}[C_{u\sigma\sigma\bar{c}}\bar{r}(\bar{r}^2 - 3\bar{s}^2) + C_{u\sigma u\bar{c}}(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4)] \\
& - 12\bar{s}^2C_{u\sigma u}\bar{c};\sigma[16C_{u\sigma\sigma\bar{c}}\bar{r}(\bar{r}^2 - 3\bar{s}^2) + 13C_{u\sigma u\bar{c}}(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4)] + 72\bar{s}^4C_{u}\bar{c}\bar{a}\bar{d};u[C_{\sigma\bar{c}\sigma\bar{d}}\bar{r}(\bar{r}^2 + 3\bar{s}^2) \\
& + 2C_{u\bar{c}\sigma\bar{d}}(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4) + C_{u\bar{c}ud}\bar{r}(\bar{r}^4 + 10\bar{r}^2\bar{s}^2 + 5\bar{s}^4)]\}u_{\bar{a}}. \tag{4.6}
\end{aligned}$$

C. Gravitational singular field

To $\mathcal{O}(\epsilon^4)$, the gravitational singular field is

$$\begin{aligned}
\bar{h}_{ab}^{(S)} = & \mu g_a{}^{\bar{a}} g_b{}^{\bar{b}} \left\{ \frac{u_{\bar{a}} u_{\bar{b}}}{\bar{s}} + \frac{1}{6\bar{s}^3} \left[(\bar{r}^2 - \bar{s}^2) C_{u\sigma u\sigma} u_{\bar{a}} u_{\bar{b}} - 6\bar{r}\bar{s}^2 C_{u\sigma u(\bar{a}} u_{\bar{b})} - 6\bar{s}^4 C_{\bar{a}u\bar{b}u} \right] + \frac{1}{24\bar{s}^3} \left[12\bar{s}^4 (C_{\bar{a}u\bar{b}u;\sigma} - \bar{r} C_{\bar{a}u\bar{b}u;u}) \right. \right. \\
& + 4\bar{s}^2 \left(u_{(\bar{a}} C_{\bar{b})u u\sigma;u}(\bar{r}^2 + \bar{s}^2) - \bar{r} u_{(\bar{a}} C_{\bar{b})u u\sigma;\sigma} \right) + u_{\bar{a}} u_{\bar{b}} \left(\bar{r}(\bar{r}^2 - 3\bar{s}^2) C_{u\sigma u\sigma;u} - (\bar{r}^2 - \bar{s}^2) C_{u\sigma u\sigma;\sigma} \right) \\
& \left. \left. + \frac{1}{1440\bar{s}^5} [\bar{h}_{\bar{a}\bar{b}}^{(S)}]_{(3)} + \frac{1}{6480\bar{s}^5} [\bar{h}_{\bar{a}\bar{b}}^{(S)}]_{(4)} + \mathcal{O}(\epsilon^5) \right\}, \tag{4.7}
\end{aligned}$$

where

$$\begin{aligned}
[\bar{h}_{\bar{a}\bar{b}}^{(S)}]_{(3)} = & -240C_{\bar{a}u\bar{b}u;\sigma\sigma}\bar{s}^6 + 240C_{\bar{a}u\bar{b}u;u\sigma}\bar{r}\bar{s}^6 + 480C_{\bar{a}}\bar{c}u\sigma C_{\bar{b}uu\bar{c}}\bar{r}\bar{s}^6 - 120C_{\bar{a}u\bar{u}\bar{c}}\bar{c}C_{\bar{b}u\bar{u}\bar{c}}\bar{s}^6 + 960C_{\bar{a}u\bar{b}}\bar{c}C_{u\sigma u\bar{c}}\bar{r}\bar{s}^6 \\
& + 40C_{\bar{a}u\bar{b}u;\bar{c}}\bar{s}^8 + 20C_{\bar{a}u}\bar{c}\bar{d}C_{\bar{b}u\bar{c}\bar{d}}\bar{s}^8 + 240C_{\bar{a}}\bar{c}\bar{b}\bar{d}C_{u\bar{c}ud}\bar{s}^8 + 360C_{\bar{a}uu\bar{u}}C_{\bar{b}uu\bar{c}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) \\
& + 240C_{\bar{a}u\bar{b}u}C_{u\sigma u\sigma}\bar{s}^4(\bar{r}^2 + \bar{s}^2) - 80C_{\bar{a}u\bar{b}u;uu}\bar{s}^6(3\bar{r}^2 + \bar{s}^2) - 40C_{\bar{a}uu}\bar{c}\bar{s}^6[6C_{\bar{b}u\bar{c}\bar{d}}\bar{r} + C_{\bar{b}uu\bar{c}}(3\bar{r}^2 + \bar{s}^2)] \\
& + \{120C_{\bar{b}uu\bar{u}};\sigma\bar{r}\bar{s}^4 + 240C_{\bar{b}uu\bar{u}}C_{u\sigma u\sigma}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) - 120C_{\bar{b}uu\bar{u};u\sigma}\bar{s}^4(\bar{r}^2 + \bar{s}^2) - 360C_{\bar{b}}\bar{c}u\sigma C_{u\sigma u\bar{c}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) \\
& + 120C_{\bar{b}uu\bar{u};uu}\bar{r}\bar{s}^4(\bar{r}^2 + 3\bar{s}^2) + 120C_{\bar{b}}\bar{c}\bar{d}[C_{u\bar{c}ud}\bar{r} + C_{\bar{c}u\bar{d}}]\bar{s}^6 + 120C_{\bar{b}u\bar{u}}\bar{c}[C_{u\sigma\sigma\bar{c}}\bar{r} + 2C_{u\sigma u\bar{c}}(\bar{r}^2 + \bar{s}^2)]\bar{s}^4 \\
& + 40C_{\bar{b}}\bar{c}\bar{d}[3C_{u\bar{c}\sigma\bar{d}}\bar{r} + C_{u\bar{c}ud}(\bar{r}^4 + 3\bar{r}^2\bar{s}^2)]\bar{s}^6 + 120C_{\bar{b}uu}\bar{c}[C_{u\sigma\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) + C_{u\sigma u\bar{c}}(\bar{r}^2 + 3\bar{s}^2)]\bar{s}^4\}u_{\bar{a}}
\end{aligned}$$

$$\begin{aligned}
& + \{4C_{\sigma\bar{c}\sigma\bar{d}}C_{\sigma}^{\bar{c}}\sigma^{\bar{d}}\bar{s}^4 - 5C_{\bar{c}\bar{d}\bar{e}\bar{f}}C^{\bar{c}\bar{d}\bar{e}\bar{f}}\bar{s}^8 + 12C_{u\sigma u\sigma;\sigma\sigma}\bar{s}^2(\bar{r}^2 - \bar{s}^2) + 16C_{u\sigma\sigma\bar{c}}C_{u\sigma\sigma}^{\bar{c}}\bar{s}^2(\bar{r}^2 - \bar{s}^2) \\
& - 12C_{u\sigma u\sigma;u\sigma}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) + 64C_{u\sigma u}^{\bar{c}}C_{u\sigma\sigma\bar{c}}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) + 60C_{u\sigma u\sigma}^2(\bar{r}^2 - \bar{s}^2)^2 + 8C_u^{\bar{c}}_u^{\bar{d}}C_{\sigma\bar{c}\sigma\bar{d}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) \\
& + 12C_{u\sigma u\sigma;uu}\bar{s}^2(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4) + 16C_{u\sigma u\bar{c}}C_{u\sigma u}^{\bar{c}}\bar{s}^2(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4) + 4C_{u\bar{c}u\bar{d}}C_u^{\bar{c}}_u^{\bar{d}}\bar{s}^4(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4) \\
& + 8C_u^{\bar{c}}\sigma^{\bar{d}}[2C_{\sigma\bar{c}\sigma\bar{d}}\bar{r} + C_{u\bar{d}\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2)]\bar{s}^4 + 8C_{u\bar{c}u\bar{d}}[C_u^{\bar{c}}\sigma^{\bar{d}}(\bar{r}^2 + \bar{s}^2) + 2C_u^{\bar{c}}_u^{\bar{d}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)]\bar{s}^4\}u_{\bar{a}}u_{\bar{b}}
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
[\bar{h}_{\bar{a}\bar{b}}^{(S)}]_{(4)} &= 270(C_{\bar{a}u\bar{b}u;\sigma\sigma\sigma} - C_{\bar{a}u\bar{b}u;u\sigma\sigma}\bar{r})\bar{s}^6 - 1080(C_{u\sigma u}^{\bar{c}};\sigma C_{\bar{a}u\bar{b}\bar{c}} + C_{\bar{a}}^{\bar{c}}\sigma;u\sigma;C_{\bar{b}uu\bar{c}})\bar{r}\bar{s}^6 - 2700C_{\bar{a}u\bar{b}}^{\bar{c}};\sigma\bar{r}C_{u\sigma u\bar{c}}\bar{s}^6 \\
&- 90C_{\bar{a}u\bar{b}u}^{\bar{c}\bar{c}}\bar{s}^8 + 90C_{\bar{a}u\bar{b}u}^{\bar{c}}\bar{c}\bar{r}\bar{s}^8 - 360C_u^{\bar{c}}_u^{\bar{d}};\sigma C_{\bar{a}\bar{c}\bar{b}\bar{d}}\bar{s}^8 + 720C_u^{\bar{c}}_u^{\bar{d}};u\bar{r}C_{\bar{a}\bar{c}\bar{b}\bar{d}}\bar{s}^8 + 180C_u^{\bar{c}}\sigma^{\bar{d}};\bar{d}C_{\bar{a}u\bar{b}\bar{c}}\bar{s}^8 \\
&+ 180C_u^{\bar{c}}_u^{\bar{d}};\bar{c}\bar{r}C_{\bar{a}u\bar{b}\bar{d}}\bar{s}^8 - 120C_{\bar{a}\bar{b}}^{\bar{c}\bar{d}};\sigma C_{\bar{b}u\bar{c}\bar{d}}\bar{s}^8 - 60C_{\bar{a}u\sigma}^{\bar{c}\bar{d}}\bar{r}C_{\bar{b}u\bar{c}\bar{d}}\bar{s}^8 + 60C_{\bar{a}\bar{u}}^{\bar{c}\bar{d}};u\bar{r}C_{\bar{b}u\bar{c}\bar{d}}\bar{s}^8 - 180C_{\bar{a}}^{\bar{c}\bar{d}};\bar{d}C_{\bar{b}uu\bar{c}}\bar{s}^8 \\
&- 180C_{\bar{a}}^{\bar{c}\bar{d}};\bar{d}\bar{r}C_{\bar{b}uu\bar{c}}\bar{s}^8 - 720C_{\bar{a}}^{\bar{c}\bar{d}};\sigma C_{u\bar{c}u\bar{d}}\bar{s}^8 + 360C_{\bar{a}}^{\bar{c}\bar{d}};\bar{d};u\bar{r}C_{u\bar{c}u\bar{d}}\bar{s}^8 - 270C_{u\sigma u\sigma;\sigma}C_{\bar{a}u\bar{b}u}\bar{s}^4(\bar{r}^2 + \bar{s}^2) \\
&- 1080C_{\bar{a}uu\sigma;\sigma}C_{\bar{b}uu\sigma}^{\bar{c}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) - 540C_{\bar{a}u\bar{b}u;\sigma}C_{u\sigma u\sigma}^{\bar{c}}\bar{s}^4(\bar{r}^2 + \bar{s}^2) - 270C_{\bar{a}u\bar{b}u;uuu}\bar{r}\bar{s}^6(\bar{r}^2 + \bar{s}^2) \\
&+ 270(C_{u\sigma u\sigma;u}C_{\bar{a}u\bar{b}u} + 4C_{\bar{a}uu\sigma;u}C_{\bar{b}uu\sigma} + 2C_{\bar{a}u\bar{b}u;u}C_{u\sigma u\sigma})\bar{r}\bar{s}^4(\bar{r}^2 + 3\bar{s}^2) + 540C_{\bar{a}u\bar{b}}^{\bar{c}};\sigma(C_{\bar{b}uu\bar{c}}\bar{r} + C_{\bar{b}u\sigma\bar{c}})\bar{s}^6 \\
&+ 90(C_{\bar{a}u\bar{b}u;uu\sigma} + 6C_{u\sigma u}^{\bar{c}};uC_{\bar{a}u\bar{b}\bar{c}} + 2C_{\bar{a}}^{\bar{c}}\sigma;u\bar{r}C_{\bar{b}uu\bar{c}} + 6C_{\bar{a}u\bar{b}}^{\bar{c}};uC_{u\sigma u\bar{c}})(3\bar{r}^2 + \bar{s}^2)\bar{s}^6 + 60C_{\bar{a}uu}^{\bar{c}\bar{d}}(3C_{\bar{b}\bar{c}\bar{u}\bar{d}}\bar{r} \\
&+ 3C_{\bar{b}\bar{c}\bar{u}\bar{d}}\bar{r})\bar{s}^8 + 180C_{\bar{a}\bar{u}\bar{b}}^{\bar{c}\bar{d}};\bar{d}(C_{u\bar{c}u\bar{d}}\bar{r} + C_{u\bar{c}\sigma\bar{d}})\bar{s}^8 - 180C_{\bar{a}\sigma u}^{\bar{c}};\bar{u}[3C_{\bar{b}\bar{u}\sigma\bar{c}}\bar{r} + C_{\bar{b}\bar{u}\bar{u}\bar{c}}(3\bar{r}^2 + \bar{s}^2)]\bar{s}^6 \\
&+ 180C_{\bar{a}uu}^{\bar{c}};\sigma[-3\bar{r}C_{\bar{b}\bar{c}\bar{u}\sigma} + 3\bar{r}C_{\bar{b}\bar{u}\sigma\bar{c}} + C_{\bar{b}\bar{u}\bar{u}\bar{c}}(3\bar{r}^2 + \bar{s}^2)]\bar{s}^6 + 90C_{\bar{a}u\bar{b}u}^{\bar{c}\bar{c}}[3C_{u\sigma u\bar{c}}\bar{r} + 2C_{u\sigma u\bar{c}}(3\bar{r}^2 + \bar{s}^2)]\bar{s}^6 \\
&+ 180C_{\bar{a}uu}^{\bar{c}};\bar{u}[-3C_{\bar{b}\bar{u}\bar{u}\bar{c}}\bar{r}(\bar{r}^2 + \bar{s}^2) + 3C_{\bar{b}\bar{c}\bar{u}\sigma}(\bar{r}^2 + \bar{s}^2) - C_{\bar{b}\bar{u}\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2)]\bar{s}^6 \\
&+ \{ - 108C_{\bar{b}uu\sigma;\sigma\sigma\sigma}\bar{r}\bar{s}^4 + 36C_u^{\bar{c}}\sigma^{\bar{d}};\bar{e}C_{\bar{b}\bar{c}\bar{d}\bar{e}}\bar{s}^8 + 36C_u^{\bar{c}}_u^{\bar{d}};\bar{e}\bar{r}C_{\bar{b}\bar{c}\bar{d}\bar{e}}\bar{s}^8 + 12C_{\bar{b}}^{\bar{c}\bar{d}\bar{e}};\sigma C_{u\bar{c}\bar{d}\bar{e}}\bar{s}^8 + 24C_{\bar{b}}^{\bar{c}}\sigma^{\bar{d}};\bar{e}C_{u\bar{c}\bar{d}\bar{e}}\bar{s}^8 \\
&+ 12C_{\bar{b}}^{\bar{c}\bar{d}\bar{e}};\bar{u}C_{u\bar{c}\bar{d}\bar{e}}\bar{r}\bar{s}^8 + 24C_{\bar{b}}^{\bar{c}}_u^{\bar{d}};\bar{e}C_{u\bar{c}\bar{d}\bar{e}}\bar{r}\bar{s}^8 - 90(3C_{u\sigma u\sigma;\sigma}C_{\bar{b}uu\sigma} + 4C_{\bar{b}uu\sigma;\sigma}C_{u\sigma u\sigma})\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) \\
&+ 108(C_{\bar{b}uu\sigma;u\sigma\sigma} + 7C_{\bar{b}}^{\bar{c}}\sigma;u\sigma;C_{u\sigma u\sigma})\bar{s}^4(\bar{r}^2 + \bar{s}^2) - 108(C_{\bar{b}uu\sigma;uu\sigma} + 4C_{\bar{b}}^{\bar{c}}\sigma;u\sigma;C_{u\sigma u\sigma})\bar{r}\bar{s}^4(\bar{r}^2 + 3\bar{s}^2) \\
&+ 90(3C_{u\sigma u\sigma;u}C_{\bar{b}uu\sigma} + 4C_{\bar{b}uu\sigma;u}C_{u\sigma u\sigma})\bar{s}^2(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4) + 108C_{\bar{b}uu\sigma;uuu}\bar{s}^4(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4) \\
&- 216C_u^{\bar{c}}\sigma^{\bar{d}};\sigma(C_{\bar{b}\bar{c}\bar{u}\bar{d}}\bar{r} + C_{\bar{b}\bar{c}\sigma\bar{d}})\bar{s}^6 - 324C_{\bar{b}}^{\bar{c}}\sigma^{\bar{d}};\sigma(C_{u\bar{c}u\bar{d}}\bar{r} + C_{u\bar{c}\sigma\bar{d}})\bar{s}^6 + 8C_{\bar{b}u}^{\bar{c}\bar{d}\bar{e}};\bar{e}(C_{u\bar{c}\bar{d}\bar{e}}\bar{r} - C_{u\bar{e}\bar{c}\bar{d}}\bar{r} \\
&+ C_{\sigma\bar{c}\bar{d}\bar{e}} - C_{\sigma\bar{c}\bar{c}\bar{d}})\bar{s}^8 - 216C_{u\sigma\sigma}^{\bar{c}};\sigma[C_{\bar{b}\bar{u}\sigma\bar{c}}\bar{r} + C_{\bar{b}\bar{u}\bar{u}\bar{c}}(\bar{r}^2 + \bar{s}^2)]\bar{s}^4 - 252C_{\bar{b}\bar{u}\sigma}^{\bar{c}};\sigma[\bar{r}C_{u\sigma\sigma\bar{c}} + 2C_{u\sigma u\bar{c}}(\bar{r}^2 + \bar{s}^2)]\bar{s}^4 \\
&+ 54C_{u\sigma u}^{\bar{c}};\sigma[6C_{\bar{b}\bar{c}\bar{u}\sigma}(\bar{r}^2 + \bar{s}^2) - 5C_{\bar{b}\bar{u}\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) - 3\bar{r}C_{\bar{b}\bar{u}\bar{u}\bar{c}}(\bar{r}^2 + 3\bar{s}^2)]\bar{s}^4 + 162C_{u\sigma\sigma}^{\bar{c}};\bar{u}[C_{\bar{b}\bar{u}\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) \\
&+ C_{\bar{b}\bar{u}\bar{u}\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)]\bar{s}^4 + 12C_{\bar{b}uu}^{\bar{c}};\sigma[-9C_{u\sigma\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) - 17\bar{r}C_{u\sigma u\bar{c}}(\bar{r}^2 + 3\bar{s}^2)]\bar{s}^4 - 48C_{\bar{b}uu\sigma}^{\bar{c}\bar{c}}[3C_{u\sigma\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) \\
&+ 4\bar{r}C_{u\sigma u\bar{c}}(\bar{r}^2 + 3\bar{s}^2)]\bar{s}^4 + 48C_{\bar{b}u\sigma}^{\bar{c}};\bar{u}[3C_{u\sigma\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2) + 8C_{u\sigma u\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)]\bar{s}^4 + 108C_{u\sigma u}^{\bar{c}};\bar{u}[3\bar{r}C_{\bar{b}\bar{u}\sigma\bar{c}}(\bar{r}^2 + 3\bar{s}^2) \\
&- 6C_{\bar{b}\bar{u}\sigma\bar{r}}(\bar{r}^2 + 3\bar{s}^2) + 2C_{\bar{b}\bar{u}\bar{u}\bar{c}}(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4)]\bar{s}^4 + 36C_{\bar{b}uu}^{\bar{c}};\bar{u}[8C_{u\sigma\sigma\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2) + 7C_{u\sigma u\bar{c}}(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4)]\bar{s}^4 \\
&- 72C_u^{\bar{c}}_u^{\bar{d}};\sigma[3C_{\bar{b}\bar{c}\bar{u}\bar{d}}\bar{r} + C_{\bar{b}\bar{c}\bar{u}\bar{d}}(\bar{r}^2 + \bar{s}^2)]\bar{s}^6 + 108C_u^{\bar{c}}\sigma^{\bar{d}};\bar{u}[3C_{\bar{b}\bar{c}\bar{u}\bar{d}}\bar{r} + C_{\bar{b}\bar{c}\bar{u}\bar{d}}(\bar{r}^2 + \bar{s}^2)]\bar{s}^6 \\
&+ 108C_u^{\bar{c}}_u^{\bar{d}};\bar{u}[3C_{\bar{b}\bar{c}\bar{u}\bar{d}}\bar{r}(\bar{r}^2 + \bar{s}^2) + C_{\bar{b}\bar{c}\bar{u}\bar{d}}(\bar{r}^2 + \bar{s}^2)]\bar{s}^6 - 108C_{\bar{b}}^{\bar{c}}_u^{\bar{d}};\sigma[3C_{u\bar{c}\bar{u}\bar{d}}\bar{r} + C_{u\bar{c}\bar{u}\bar{d}}(\bar{r}^2 + \bar{s}^2)]\bar{s}^6 \\
&+ 24C_{\bar{b}u\sigma}^{\bar{c}\bar{d}}[3C_{u\bar{c}\bar{u}\bar{d}}\bar{r} + 3C_{u\bar{d}\sigma\bar{c}}\bar{r} + 3C_{\bar{c}\bar{c}\sigma\bar{d}} + C_{u\bar{c}\bar{u}\bar{d}}(\bar{r}^2 + \bar{s}^2)]\bar{s}^6 + 72C_{\bar{b}}^{\bar{c}}\sigma^{\bar{d}};\bar{u}[3\bar{r}C_{u\bar{c}\bar{u}\bar{d}} + C_{u\bar{c}\bar{u}\bar{d}}(\bar{r}^2 + \bar{s}^2)]\bar{s}^6 \\
&+ 72C_{\bar{b}}^{\bar{c}}_u^{\bar{d}};\bar{u}[3C_{u\bar{c}\bar{u}\bar{d}}\bar{r}(\bar{r}^2 + \bar{s}^2) + C_{u\bar{c}\bar{u}\bar{d}}(\bar{r}^2 + \bar{s}^2)]\bar{s}^6 \\
&+ 24C_{\bar{b}uu}^{\bar{c}\bar{d}}[3C_{u\bar{c}\bar{c}\bar{d}}\bar{r} + 3C_{u\bar{c}\bar{u}\bar{d}}\bar{r}(\bar{r}^2 + \bar{s}^2) + C_{u\bar{c}\bar{u}\bar{d}}(\bar{r}^2 + \bar{s}^2) + C_{u\bar{d}\sigma\bar{c}}(\bar{r}^2 + \bar{s}^2)]\bar{s}^6\}u_{\bar{a}} \\
&+ \{ - 18(C_{\sigma}^{\bar{c}}\sigma^{\bar{d}};\sigma - C_{\sigma}^{\bar{c}}\sigma^{\bar{d}};\bar{u}\bar{r})C_{\sigma\bar{c}\sigma\bar{d}}\bar{s}^4 + 27C^{\bar{c}\bar{d}\bar{e}\bar{f}}\sigma C_{\bar{c}\bar{d}\bar{e}\bar{f}}\bar{s}^8 - 18C^{\bar{c}\bar{d}\bar{e}\bar{f}}\bar{r}C_{\bar{c}\bar{d}\bar{e}\bar{f}}\bar{s}^8 + 9C_{u\sigma u\sigma;u\sigma\sigma}\bar{r}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) \\
&- 15C_{u\sigma u\sigma}^{\bar{c}};\bar{r}C_{u\sigma\sigma\bar{c}}\bar{s}^2(\bar{r}^2 - 3\bar{s}^2) - 135C_{u\sigma u\sigma;\sigma}C_{u\sigma u\sigma}(\bar{r}^2 - \bar{s}^2)^2 + 9C_{u\sigma u\sigma;uuu}\bar{r}\bar{s}^2(\bar{r}^4 - 10\bar{r}^2\bar{s}^2 - 15\bar{s}^4) \\
&+ 9C_{u\sigma u\sigma;u\bar{u}\bar{u}}\bar{s}^2(-\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + 3\bar{s}^4) + 15C_{u\sigma u\sigma}^{\bar{c}}C_{u\sigma u\bar{c}}\bar{s}^2(-\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + 3\bar{s}^4) + C_{u\sigma u\sigma;\sigma\sigma\sigma}(-9\bar{r}^2\bar{s}^2 + 9\bar{s}^4) \\
&+ 45C_{u\sigma u\sigma;u}C_{u\sigma u\sigma}(3\bar{r}^5 - 10\bar{r}^3\bar{s}^2 + 15\bar{r}\bar{s}^4) - 54C_{u\sigma u\sigma}^{\bar{c}};\sigma[2C_{u\sigma u\bar{c}}\bar{r}(\bar{r}^2 - 3\bar{s}^2) + C_{u\sigma u\bar{c}}(\bar{r}^2 - \bar{s}^2)]\bar{s}^2 \\
&+ 36C_{u\bar{c}\bar{u}\bar{d}}\bar{s}^4[-C_{\sigma}^{\bar{c}}\sigma^{\bar{d}};\bar{u}\bar{r} + C_{\sigma}^{\bar{c}}\sigma^{\bar{d}};\bar{u}(\bar{r}^2 + \bar{s}^2)] - 18C_{u\bar{c}\bar{u}\bar{d}}\bar{s}^4[C_{\sigma}^{\bar{c}}\sigma^{\bar{d}};\sigma(\bar{r}^2 + \bar{s}^2) - C_{\sigma}^{\bar{c}}\sigma^{\bar{d}};\bar{u}\bar{r}(\bar{r}^2 + 3\bar{s}^2)] \\
&- 36C_u^{\bar{c}}_u^{\bar{d}};\sigma[C_{\sigma\bar{c}\sigma\bar{d}}\bar{r} + C_{\bar{c}\bar{c}\sigma\bar{d}}\bar{r}(\bar{r}^2 + \bar{s}^2) + C_{u\bar{d}\sigma\bar{c}}\bar{r}(\bar{r}^2 + \bar{s}^2) + C_{u\bar{c}\bar{u}\bar{d}}\bar{r}(\bar{r}^2 + 3\bar{s}^2)]\bar{s}^4 + 15C_{u\sigma u}^{\bar{c}};\bar{u}[2C_{u\sigma\sigma\bar{c}}\bar{r}(\bar{r}^2 - 3\bar{s}^2) \\
&+ 5C_{u\sigma u\bar{c}}(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4)]\bar{s}^2 + 18C_{u\sigma u}^{\bar{c}};\bar{u}[3\bar{r}C_{u\sigma u\bar{c}}(\bar{r}^4 - 10\bar{r}^2\bar{s}^2 - 15\bar{s}^4) + 5C_{u\sigma u\bar{c}}(\bar{r}^4 - 6\bar{r}^2\bar{s}^2 - 3\bar{s}^4)]\bar{s}^2 \\
&- 18C_u^{\bar{c}}_u^{\bar{d}};\sigma[C_{\sigma\bar{c}\sigma\bar{d}}\bar{r}(\bar{r}^2 + \bar{s}^2) + 2C_{u\bar{c}\bar{u}\bar{d}}\bar{r}(\bar{r}^2 + 3\bar{s}^2) + 18C_{u\bar{c}\bar{u}\bar{d}}(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4)]\bar{s}^4 \\
&+ 36C_u^{\bar{c}}\sigma^{\bar{d}};\bar{u}[C_{\sigma\bar{c}\sigma\bar{d}}\bar{r}(\bar{r}^2 + \bar{s}^2) + C_{u\bar{c}\sigma\bar{d}}\bar{r}(\bar{r}^2 + 3\bar{s}^2) + C_{u\bar{d}\sigma\bar{c}}\bar{r}(\bar{r}^2 + 3\bar{s}^2) + C_{u\bar{c}\bar{u}\bar{d}}\bar{r}(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4)]\bar{s}^4 \\
&+ C_{u\sigma u}^{\bar{c}};\sigma[-48C_{u\sigma u\bar{c}}\bar{r}(\bar{r}^2 - 3\bar{s}^2) + 39C_{u\sigma u\bar{c}}(-\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + 3\bar{s}^4)]\bar{s}^2 + 18C_u^{\bar{c}}_u^{\bar{d}};\bar{u}[C_{\sigma\bar{c}\sigma\bar{d}}\bar{r}(\bar{r}^2 + 3\bar{s}^2) \\
&+ 2C_{u\bar{c}\sigma\bar{d}}(\bar{r}^4 + 6\bar{r}^2\bar{s}^2 + \bar{s}^4) + C_{u\bar{c}\bar{u}\bar{d}}\bar{r}(\bar{r}^4 + 10\bar{r}^2\bar{s}^2 + 5\bar{s}^4)]\bar{s}^4\}u_{\bar{a}}u_{\bar{b}}
\end{aligned} \tag{4.10}$$

V. MODE-SUM REGULARIZATION PARAMETERS

The singular field expansions derived in the previous sections have several applications in explicit self-force calculations. One of the most successful computational approaches to date is the *mode-sum* scheme of Barack and Ori [35, 36]; the majority of existing calculations are based on it in one form or another [50–73]. The basic idea is to decompose the singular retarded field into spherical harmonic modes which are continuous and finite in general for the scalar case and in Lorenz gauge for the electromagnetic and gravitational cases. A key component of the calculation involves the subtraction of so-called *regularization parameters* - analytically derived expressions which render the formally divergent sum over spherical harmonic modes finite. In this section, we derive these parameters from our singular field expressions and show how they may be used to compute the self-force with extremely high accuracy.

A. Rotated Coordinates

In order to obtain expressions which are readily written as mode-sums, previous calculations [35, 50, 53] found it useful to work in a rotated coordinate frame. We found it most efficient to carry out this rotation prior to doing any calculations. To this end, we introduce Riemann normal coordinates on the 2-sphere at \bar{x} in the form

$$w_1 = 2 \sin\left(\frac{\alpha}{2}\right) \cos\beta, \quad w_2 = 2 \sin\left(\frac{\alpha}{2}\right) \sin\beta, \quad (5.1)$$

where α and β are rotated angular coordinates given by

$$\sin\theta \cos\phi = \cos\alpha, \quad (5.2)$$

$$\sin\theta \sin\phi = \sin\alpha \cos\beta, \quad (5.3)$$

$$\cos\theta = \sin\alpha \sin\beta. \quad (5.4)$$

In these coordinates, the Schwarzschild metric is given by the line element

$$ds^2 = -\left(\frac{r-2m}{r}\right)dt^2 + \left(\frac{r}{r-2m}\right)dr^2 + r^2 \left\{ \left[\frac{16-w_2^2(8-w_1^2-w_2^2)}{4(4-w_1^2-w_2^2)} \right] dw_1^2 + 2dw_1dw_2 \left[\frac{w_1w_2(8-w_1^2-w_2^2)}{4(4-w_1^2-w_2^2)} \right] + \left[\frac{16-w_1^2(8-w_1^2-w_2^2)}{4(4-w_1^2-w_2^2)} \right] dw_2^2 \right\}. \quad (5.5)$$

The algebraic form of the metric makes it very suitable for using with computer algebra programmes such as Mathematica. The apparent complexity of having a non-diagonal metric on S^2 is in fact minimal since the determinant of that metric is simply 1.

B. Mode decomposition

The method of regularization of the self force through l -mode decomposition is by now standard, see, for example, [35], [53] and [50]. Having calculated the singular field, it is straightforward to calculate the singular self-force¹, F_a , for scalar, electromagnetic and gravitational cases using Eqs. (2.14), (2.19) and (2.24) respectively. We study the multipole decomposition of F_a by writing

$$F_a(r, t, \alpha, \beta) = \sum_{lm} F_a^{lm}(r, t) Y^{lm}(\alpha, \beta), \quad (5.6)$$

where $Y^{lm}(\theta, \phi)$ are scalar spherical harmonics, and accordingly

$$F_a^{lm}(r, t) = \int F_a(r, t, \alpha, \beta) Y^{lm*}(\alpha, \beta) d\Omega. \quad (5.7)$$

¹ In this section, for notational convenience we drop the implied (S) superscript denoting “singular” as we are always referring to the singular component.

To calculate the l -mode contribution at $\bar{x} = (t_0, r_0, \alpha_0, \beta_0)$, we have

$$F_a^l(r_0, t_0) = \lim_{\Delta r \rightarrow 0} \sum_m F_a^{lm}(r_0 + \Delta r, t_0) Y^{lm}(\alpha_0, \beta_0). \quad (5.8)$$

In previous calculations, Eq. (5.6) has naturally arisen in Schwarzschild coordinates with $\theta_0 = \frac{\pi}{2}$, and it was necessary to perform a rotation to move the coordinate location of the particle from the equatorial plane to a pole in the new coordinate system. However, by choosing to work in an S^2 Riemann normal coordinate system from the start, our particle is already located on the pole. This saves us from further transformation and expansions at this stage. With the particle on the pole, $Y^{lm}(\alpha_0 = 0, \beta_0) = 0$ for all $m \neq 0$. This also allows us, without loss of generality, to take $\beta_0 = 0$. Taking α_0, β_0 and m all to be equal to zero in Eq. (5.8) gives us

$$F_a^l(r_0, t_0) = \lim_{\Delta r \rightarrow 0} \sqrt{\frac{2l+1}{4\pi}} F_a^{l,m=0}(r_0 + \Delta r, t_0) = \frac{2l+1}{4\pi} \lim_{\Delta r \rightarrow 0} \int F_a(r_0 + \Delta r, t_0, \alpha, \beta) P_l(\cos \alpha) d\Omega. \quad (5.9)$$

For each spin field, the singular self-force, $F_a(r, t, \alpha, \beta)$, has the form

$$F_a(r, t, \alpha, \beta) = \sum_{n=1} \frac{B_a^{(3n-2)}}{\rho^{2n+1}} \epsilon^{n-3}, \quad (5.10)$$

where $B_a^{(k)} = b_{a_1 a_2 \dots a_k}(\bar{x}) \Delta x^{a_1} \Delta x^{a_2} \dots \Delta x^{a_k}$. On identifying $\tau_1 = \bar{r}_{(1)} \pm \rho$, this form can be easily seen to follow from the coordinate representation of the above expressions for the singular field. In using Eq. (5.10) to determine the regularization parameters, we only need to take the sum to the appropriate order: $n = 1$ for A_a , $n = 2$ for B_a , etc.

Explicitly, in our coordinates $\rho = \sqrt{(g_{\bar{a}\bar{b}} u^{\bar{a}} \Delta x^b)^2 + g_{\bar{a}\bar{b}} \Delta x^a \Delta x^b}$ takes the form

$$\begin{aligned} \rho(r, t, \alpha, \beta)^2 &= \frac{(E^2 r_0^3 - L^2(r_0 - 2M))}{r_0(r_0 - 2M)^2} \Delta r^2 + (L^2 + r_0^2) \Delta w_1^2 - \left(\frac{2Er_0 \dot{r}_0}{r_0 - 2M} \Delta r + 2EL \Delta w_1 \right) \Delta t + \frac{2Lr_0 \dot{r}_0}{r_0 - 2M} \Delta r \Delta w_1 \\ &\quad + \left(E^2 + \frac{2M}{r_0} - 1 \right) \Delta t^2 + r_0^2 \Delta w_2^2, \end{aligned} \quad (5.11)$$

where the α, β dependence is contained exclusively in Δw_1 and Δw_2 . In particular, taking $t = t_0$ ($\Delta t = 0$) allows us to write

$$\rho(r, t_0, \alpha, \beta)^2 = \frac{E^2 r_0^4}{(L^2 + r_0^2)(r_0 - 2M)^2} \Delta r^2 + (L^2 + r_0^2) \left(\Delta w_1 + \frac{Lr_0 \dot{r}_0}{(r_0 - 2M)(L^2 + r_0^2)} \Delta r \right)^2 + r_0^2 \Delta w_2^2. \quad (5.12)$$

We further define

$$\rho_0(\alpha, \beta)^2 = \rho(r_0, t_0, \alpha, \beta)^2 = (L^2 + r_0^2) \Delta w_1^2 + r_0^2 \Delta w_2^2 \quad (5.13)$$

and this allows us to rewrite our Δw 's in the alternate forms

$$\Delta w_1^2 = 2(1 - \cos \alpha) \cos^2 \beta = \frac{\rho_0^2}{(r_0^2 + L^2)\chi} \cos^2 \beta = \frac{\rho_0^2}{L^2 \chi} (k - (1 - \chi)), \quad (5.14)$$

$$\Delta w_2^2 = 2(1 - \cos \alpha) \sin^2 \beta = \frac{\rho_0^2}{(r_0^2 + L^2)\chi} \sin^2 \beta = \frac{\rho_0^2}{L^2 \chi} (1 - \chi), \quad (5.15)$$

where

$$\chi(\beta) = 1 - k \sin^2 \beta, \quad k = \frac{L^2}{r_0^2 + L^2}. \quad (5.16)$$

Suppose, for the moment, that we may take the limit in Eq. (5.9) through the integral sign, then using our alternate forms we have

$$\lim_{\Delta r \rightarrow 0} \frac{B_a^{(3n-2)}}{\rho^{2n+1}} \epsilon^{n-3} = \frac{b_{i_1 i_2 \dots i_{3n-2}}(r_0) \Delta w^{i_1} \Delta w^{i_2} \dots \Delta w^{i_{3n-2}}}{\rho_0^{2n+1}} \epsilon^{n-3} = \rho_0^{n-3} \epsilon^{n-3} c_{a(n)}(r_0, \chi). \quad (5.17)$$

In [74], it was shown that the integral and limit in Eq.(5.9) are indeed interchangeable for all orders except the leading order, $n = 1$ term, where the limiting ρ_0^{-3} would not be integrable. Thus we find the singular self-force now has the form

$$\begin{aligned} F_a^l(r_0, t_0, \alpha, \beta) &= \frac{2l+1}{4\pi} \left[\epsilon^{-2} \lim_{\Delta r \rightarrow 0} \int \frac{B_a^{(1)}}{\rho^3} P_l(\cos \alpha) d\Omega + \sum_{n=2} \epsilon^{n-3} \int \rho_0^{n-3} c_{a(n)}(r_0, \chi) P_l(\cos \alpha) d\Omega \right] \\ &\equiv F_{a[-1]}^l(r_0, t_0) \epsilon^{-2} + F_{a[0]}^l(r_0, t_0) \epsilon^{-1} + F_{a[2]}^l(r_0, t_0) \epsilon^1 + F_{a[4]}^l(r_0, t_0) \epsilon^3 + F_{a[6]}^l(r_0, t_0) \epsilon^5 + \dots, \end{aligned} \quad (5.18)$$

where the β dependence in the c_n 's are hidden in χ , while the α and β dependence of $F_a(r, t_0, \alpha, \beta)$ is hidden in both the ρ 's and c_n 's. Note here that we use the convention that a subscript in square brackets denotes the term which will contribute at that order in $1/l$. Furthermore the integrand in the summation is odd or even under $\Delta w_i \rightarrow -\Delta w_i$ according to whether n (and so $3n-2$) is odd or even. This means only the even terms are non-vanishing, while $F_{a[1]}^l(r_0, t_0) = F_{a[3]}^l(r_0, t_0) = F_{a[5]}^l(r_0, t_0) = 0$ etc.

Some care is required in dealing with taking the limit in the first term. As this has been addressed previously [50, 53, 74, 75] and our main interest is in the higher orders we omit the details here and only discuss the calculation of the higher order terms.

In the higher order terms in Eq. (5.18), we may immediately work with $\rho_0^2 = 2\chi(L^2 + r_0^2)(1 - \cos \alpha)$ so,

$$\rho_0(r_0, t_0, \alpha, \beta)^n = [2\chi(L^2 + r_0^2)(1 - \cos \alpha)]^{n/2} = [2\chi(L^2 + r_0^2)]^{n/2} \sum_{l=0} \mathcal{A}_l^{n/2}(0) P_l(\cos \alpha), \quad (5.19)$$

where $\mathcal{A}_l^{-\frac{1}{2}}(0) = \sqrt{2}$ from the generating function of the Legendre polynomials and, as derived in Appendix D of [53], for $(n+1)/2 \in \mathbb{N}$

$$\begin{aligned} \mathcal{P}_{n/2} &= (-1)^{(n+1)/2} 2^{1+n/2} (n!!)^2, \\ \mathcal{A}_l^{n/2}(0) &= \frac{\mathcal{P}_{n/2}(2l+1)}{(2l-n)(2l-n+2)\dots(2l+n)(2l+n+2)}. \end{aligned} \quad (5.20)$$

In this case the angular integrals involve

$$\frac{1}{2\pi} \int \frac{d\beta}{\chi(\beta)^{n/2}} = \langle \chi^{-n/2}(\beta) \rangle = {}_2F_1\left(\frac{n}{2}, \frac{1}{2}, 1, k\right) \quad (5.21)$$

where $(n+1)/2 \in \mathbb{N} \cup \{0\}$. The resulting equations can then be tidied up using the following special cases of hypergeometric functions

$$\langle \chi^{-\frac{1}{2}} \rangle = \mathcal{F}_{\frac{1}{2}}(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k\right) = \frac{2}{\pi} \mathcal{K}, \quad (5.22)$$

$$\langle \chi^{\frac{1}{2}} \rangle = \mathcal{F}_{-\frac{1}{2}}(k) = {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; k\right) = \frac{2}{\pi} \mathcal{E}, \quad (5.23)$$

where

$$\mathcal{K} \equiv \int_0^{\pi/2} (1 - k \sin^2 \beta)^{-1/2} d\beta, \quad \mathcal{E} \equiv \int_0^{\pi/2} (1 - k \sin^2 \beta)^{1/2} d\beta \quad (5.24)$$

are complete elliptic integrals of the first and second kinds, respectively. All other powers of χ can be integrated to hypergeometric functions that can then be manipulated to be one of the above by the use of the recurrence relation in Eq. (15.2.10) of [76]. That is

$$\mathcal{F}_{p+1}(k) = \frac{p-1}{p(k-1)} \mathcal{F}_{p-1}(k) + \frac{1-2p+(p-\frac{1}{2})k}{p(k-1)} \mathcal{F}_p(k). \quad (5.25)$$

In the next sections, we give the results of applying this calculation to each of scalar, electromagnetic and gravitational cases in turn. In doing so, we omit the explicit dependence on l which in each case is

$$\begin{aligned} F_{a[-1]}^l &= (2l+1) F_{a[-1]}, \quad F_{a[0]}^l = F_{a[0]}, \quad F_{a[2]}^l = \frac{F_{a[2]}}{(2l-1)(2l+3)}, \quad F_{a[4]}^l = \frac{F_{a[4]}}{(2l-3)(2l-1)(2l+3)(2l+5)}, \\ F_{a[6]}^l &= \frac{F_{a[6]}}{(2l-5)(2l-3)(2l-1)(2l+3)(2l+5)(2l+7)}. \end{aligned} \quad (5.26)$$

It is also worth pointing out that there exists in the literature several different notations for the regularization parameters. We have adopted a notation which is readily extensible to other orders and which makes the dependence on l explicit. To avoid confusion, in Table I we give the relation between our notation and other common notations.

RP	BO	DMW
$F_{a[-1]}$	A_a	A_a
$F_{a[0]}$	B_a	B_a
$F_{a[1]}$	C_a	C_a
$F_{a[2]}$	—	D_a
$F_{a[4]}$	—	E_a^1
$F_{a[6]}$	—	E_a^2

TABLE I. Relation between notational choices for the regularization parameters (RPs). The most common choices are those of either Barack and Ori [35] or Detweiler, Messaritaki and Whiting [53].

C. Scalar case

In the scalar case, the regularization parameters are given by

$$F_{t[-1]} = \frac{\dot{r}_0 \operatorname{sgn}(\Delta r)}{2(L^2 + r_0^2)}, \quad F_{r[-1]} = -\frac{Er_0 \operatorname{sgn}(\Delta r)}{2(r_0 - 2M)(L^2 + r_0^2)}, \quad F_{\theta[-1]} = 0, \quad F_{\phi[-1]} = 0, \quad (5.27)$$

$$F_{t[0]} = -\frac{Er_0 \dot{r}_0}{\pi(L^2 + r_0^2)^{3/2}}(2\mathcal{E} - \mathcal{K}), \quad (5.28)$$

$$F_{r[0]} = \frac{1}{\pi r_0 (r_0 - 2M)(L^2 + r_0^2)^{3/2}} \left\{ [2E^2 r_0^3 - (r_0 - 2M)(L^2 + r_0^2)]\mathcal{E} - [E^2 r_0^3 + (r_0 - 2M)(L^2 + r_0^2)]\mathcal{K} \right\},$$

$$F_{\theta[0]} = 0, \quad F_{\phi[0]} = -\frac{r_0 \dot{r}_0}{L\pi(L^2 + r_0^2)^{1/2}}(\mathcal{E} - \mathcal{K}), \quad (5.29)$$

$$F_{t[2]} = \frac{E\dot{r}_0}{2\pi r_0^4 (L^2 + r_0^2)^{7/2}}(F_{t[2]}^{\mathcal{E}}\mathcal{E} + F_{t[2]}^{\mathcal{K}}\mathcal{K}), \quad (5.30)$$

where

$$F_{t[2]}^{\mathcal{E}} = 8E^2(L^2 - r_0^2)r_0^7 - (L^2 + r_0^2)(36L^6M + 104L^4Mr_0^2 + 98L^2Mr_0^4 + L^2r_0^5 + 46Mr_0^6 - 7r_0^7),$$

$$F_{t[2]}^{\mathcal{K}} = -E^2r_0^7(3L^2 - 5r_0^2) + 2r_0^2(L^2 + r_0^2)(9L^4M + 18L^2Mr_0^2 + 13Mr_0^4 - 2r_0^5),$$

$$F_{r[2]} = \frac{1}{2\pi r_0^6 (r_0 - 2M)(L^2 + r_0^2)^{7/2}}(F_{r[2]}^{\mathcal{E}}\mathcal{E} + F_{r[2]}^{\mathcal{K}}\mathcal{K}), \quad (5.31)$$

where

$$F_{r[2]}^{\mathcal{E}} = -8E^4r_0^{10}(L^2 - r_0^2) + 4E^2r_0^3(L^2 + r_0^2)(9L^6M + 26L^4Mr_0^2 + 23L^2Mr_0^4 + L^2r_0^5 + 14Mr_0^6 - 3r_0^7) - (r_0 - 2M)(L^2 + r_0^2)^2(28L^6M + 82L^4Mr_0^2 + 82L^2Mr_0^4 - L^2r_0^5 + 32Mr_0^6 - 3r_0^7),$$

$$F_{r[2]}^{\mathcal{K}} = E^4r_0^{10}(3L^2 - 5r_0^2) - E^2r_0^5(L^2 + r_0^2)(18L^4M + 34L^2Mr_0^2 + L^2r_0^3 + 32Mr_0^4 - 7r_0^5) + (r_0 - 2M)r_0^2(L^2 + r_0^2)^2(14L^4M + 28L^2Mr_0^2 + 16Mr_0^4 - r_0^5),$$

$$F_{\theta[2]} = 0, \quad (5.32)$$

$$F_{\phi[2]} = \frac{\dot{r}_0}{2\pi L r_0^4 (L^2 + r_0^2)^{5/2}} (F_{\phi[2]}^{\mathcal{E}} \mathcal{E} + F_{\phi[2]}^{\mathcal{K}} \mathcal{K}), \quad (5.33)$$

where

$$\begin{aligned} F_{\phi[2]}^{\mathcal{E}} &= E^2 r_0^7 (7L^2 - r_0^2) + (L^2 + r_0^2)(28L^6 M + 58L^4 M r_0^2 + 34L^2 M r_0^4 - L^2 r_0^5 + r_0^7), \\ F_{\phi[2]}^{\mathcal{K}} &= -E^2 r_0^7 (3L^2 - r_0^2) - r_0^2 (L^2 + r_0^2)(14L^4 M + 16L^2 M r_0^2 + r_0^5), \end{aligned}$$

$$F_{t[4]} = \frac{3E\dot{r}_0}{40\pi r_0^{11} (L^2 + r_0^2)^{11/2}} (F_{t[4]}^{\mathcal{E}} \mathcal{E} + F_{t[4]}^{\mathcal{K}} \mathcal{K}), \quad (5.34)$$

where

$$\begin{aligned} F_{t[4]}^{\mathcal{E}} &= -30E^4 r_0^{16} (23L^4 - 82L^2 r_0^2 + 23r_0^4) \\ &\quad + 2E^2 r_0^5 (L^2 + r_0^2)(44800L^{12} M + 219136L^{10} M r_0^2 + 428252L^8 M r_0^4 + 418776L^6 M r_0^6 \\ &\quad + 206374L^4 M r_0^8 + 45L^4 r_0^9 + 45188L^2 M r_0^{10} - 1230L^2 r_0^{11} - 166M r_0^{12} + 645r_0^{13}) \\ &\quad - 2(L^2 + r_0^2)^2 (20480L^{14} M^2 - 97280L^{12} M^2 r_0^2 + 85120L^{12} M r_0^3 - 700832L^{10} M^2 r_0^4 \\ &\quad + 388480L^{10} M r_0^5 - 1426472L^8 M^2 r_0^6 + 704552L^8 M r_0^7 - 1358276L^6 M^2 r_0^8 \\ &\quad + 635226L^6 M r_0^9 - 635180L^4 M^2 r_0^{10} + 286498L^4 M r_0^{11} - 15L^4 r_0^{12} - 124540L^2 M^2 r_0^{12} \\ &\quad + 54086L^2 M r_0^{13} - 90L^2 r_0^{14} - 2796M^2 r_0^{14} + 182M r_0^{15} + 285r_0^{16}), \\ F_{t[4]}^{\mathcal{K}} &= 15E^4 r_0^{16} (15L^4 - 82L^2 r_0^2 + 31r_0^4) \\ &\quad - 4E^2 r_0^7 (L^2 + r_0^2)(11200L^{10} M + 44984L^8 M r_0^2 + 68227L^6 M r_0^4 \\ &\quad + 46849L^4 M r_0^6 + 13493L^2 M r_0^8 - 270L^2 r_0^9 + 127M r_0^{10} + 210r_0^{11}) \\ &\quad + r_0^2 (L^2 + r_0^2)^2 (20480L^{12} M^2 - 115200L^{10} M^2 r_0^2 + 85120L^{10} M r_0^3 - 599072L^8 M^2 r_0^4 \\ &\quad + 314000L^8 M r_0^5 - 908104L^6 M^2 r_0^6 + 433792L^6 M r_0^7 - 589164L^4 M^2 r_0^8 + 268648L^4 M r_0^9 \\ &\quad - 151484L^2 M^2 r_0^{10} + 66380L^2 M r_0^{11} - 15L^2 r_0^{12} - 5592M^2 r_0^{12} + 1204M r_0^{13} + 345r_0^{14}), \end{aligned}$$

$$F_{r[4]} = \frac{3}{40\pi r_0^{13} (r_0 - 2M) (L^2 + r_0^2)^{11/2}} (F_{r[4]}^{\mathcal{E}} \mathcal{E} + F_{r[4]}^{\mathcal{K}} \mathcal{K}), \quad (5.35)$$

where

$$\begin{aligned} F_{r[4]}^{\mathcal{E}} &= 30E^6 r_0^{19} (23L^4 - 82L^2 r_0^2 + 23r_0^4) \\ &\quad - E^4 r_0^8 (L^2 + r_0^2)(89600L^{12} M + 438272L^{10} M r_0^2 + 856504L^8 M r_0^4 + 837552L^6 M r_0^6 \\ &\quad + 411938L^4 M r_0^8 + 495L^4 r_0^9 + 92836L^2 M r_0^{10} - 3690L^2 r_0^{11} - 902M r_0^{12} + 1575r_0^{13}) \\ &\quad + 8E^2 r_0^3 (L^2 + r_0^2)^2 (5120L^{14} M^2 - 35200L^{12} M^2 r_0^2 + 26720L^{12} M r_0^3 - 227368L^{10} M^2 r_0^4 + 123200L^{10} M r_0^5 \\ &\quad - 456300L^8 M^2 r_0^6 + 225979L^8 M r_0^7 - 434510L^6 M^2 r_0^8 + 206277L^6 M r_0^9 - 203983L^4 M^2 r_0^{10} \\ &\quad + 94211L^4 M r_0^{11} - 40376L^2 M^2 r_0^{12} + 18367L^2 M r_0^{13} - 135L^2 r_0^{14} - 571M^2 r_0^{14} - 146M r_0^{15} + 135r_0^{16}) \\ &\quad - (r_0 - 2M) (L^2 + r_0^2)^3 (40960L^{14} M^2 - 86016L^{12} M^2 r_0^2 + 116480L^{12} M r_0^3 - 860224L^{10} M^2 r_0^4 \\ &\quad + 510080L^{10} M r_0^5 - 1780112L^8 M^2 r_0^6 + 882400L^8 M r_0^7 - 1657392L^6 M^2 r_0^8 + 752340L^6 M r_0^9 \\ &\quad - 743164L^4 M^2 r_0^{10} + 316100L^4 M r_0^{11} + 30L^4 r_0^{12} - 136236L^2 M^2 r_0^{12} + 53200L^2 M r_0^{13} + 75L^2 r_0^{14} \\ &\quad - 3120M^2 r_0^{14} + 160M r_0^{15} + 165r_0^{16}), \\ F_{r[4]}^{\mathcal{K}} &= -15E^6 r_0^{19} (15L^4 - 82L^2 r_0^2 + 31r_0^4) \\ &\quad + E^4 r_0^{10} (L^2 + r_0^2)(44800L^{10} M + 179936L^8 M r_0^2 + 272908L^6 M r_0^4 \\ &\quad + 187126L^4 M r_0^6 + 135L^4 r_0^7 + 55232L^2 M r_0^8 - 1710L^2 r_0^9 + 118M r_0^{10} + 1035r_0^{11}) \\ &\quad - E^2 r_0^5 (L^2 + r_0^2)^2 (20480L^{12} M^2 - 158720L^{10} M^2 r_0^2 + 106880L^{10} M r_0^3 - 769632L^8 M^2 r_0^4 \\ &\quad + 399280L^8 M r_0^5 - 1159632L^6 M^2 r_0^6 + 559556L^6 M r_0^7 - 755876L^4 M^2 r_0^8 + 352004L^4 M r_0^9 \end{aligned}$$

$$\begin{aligned}
& -196524L^2M^2r_0^{10} + 89680L^2Mr_0^{11} - 405L^2r_0^{12} - 5528M^2r_0^{12} + 512Mr_0^{13} + 675r_0^{14}) \\
& + (r_0 - 2M)r_0^2(L^2 + r_0^2)^3(20480L^{12}M^2 - 60928L^{10}M^2r_0^2 + 58240L^{10}Mr_0^3 - 375840L^8M^2r_0^4 \\
& \quad + 204080L^8Mr_0^5 - 564472L^6M^2r_0^6 + 265360L^6Mr_0^7 - 350956L^4M^2r_0^8 + 152380L^4Mr_0^9 \\
& \quad - 84276L^2M^2r_0^{10} + 33740L^2Mr_0^{11} + 15L^2r_0^{12} - 3120M^2r_0^{12} + 640Mr_0^{13} + 75r_0^{14}),
\end{aligned}$$

$$F_{\theta[4]} = 0, \quad (5.36)$$

$$F_{\phi[4]} = \frac{3\dot{r}_0}{40\pi Lr_0^{11}(L^2 + r_0^2)^{9/2}}(F_{\phi[4]}^{\mathcal{E}}\mathcal{E} + F_{\phi[4]}^{\mathcal{K}}\mathcal{K}), \quad (5.37)$$

where

$$\begin{aligned}
F_{\phi[4]}^{\mathcal{E}} &= -15E^4r_0^{16}(43L^4 - 82L^2r_0^2 + 3r_0^4) \\
&\quad - 10E^2r_0^5(L^2 + r_0^2)(4352L^{12}M + 16512L^{10}Mr_0^2 + 22948L^8Mr_0^4 \\
&\quad + 13346L^6Mr_0^6 + 2136L^4Mr_0^8 - 9L^4r_0^9 - 710L^2Mr_0^{10} + 126L^2r_0^{11} - 9r_0^{13}) \\
&\quad + (L^2 + r_0^2)^2(40960L^{14}M^2 - 96256L^{12}M^2r_0^2 + 116480L^{12}Mr_0^3 - 704064L^{10}M^2r_0^4 \\
&\quad + 429440L^{10}Mr_0^5 - 1134992L^8M^2r_0^6 + 595040L^8Mr_0^7 - 755632L^6M^2r_0^8 \\
&\quad + 372500L^6Mr_0^9 - 194724L^4M^2r_0^{10} + 94940L^4Mr_0^{11} + 30L^4r_0^{12} - 6276L^2M^2r_0^{12} \\
&\quad + 4040L^2Mr_0^{13} + 105L^2r_0^{14} + 480M^2r_0^{14} - 45r_0^{16}), \\
F_{\phi[4]}^{\mathcal{K}} &= 15E^4r_0^{16}(L^2 - 3r_0^2)(15L^2 - r_0^2) \\
&\quad + 10E^2r_0^7(L^2 + r_0^2)(2176L^{10}M + 6352L^8Mr_0^2 + 6018L^6Mr_0^4 \\
&\quad + 1666L^4Mr_0^6 - 320L^2Mr_0^8 + 63L^2r_0^9 - 9r_0^{11}) \\
&\quad - r_0^2(L^2 + r_0^2)^2(20480L^{12}M^2 - 66048L^{10}M^2r_0^2 + 58240L^{10}Mr_0^3 - 293280L^8M^2r_0^4 \\
&\quad + 163760L^8Mr_0^5 - 314392L^6M^2r_0^6 + 156960L^6Mr_0^7 - 114876L^4M^2r_0^8 \\
&\quad + 55420L^4Mr_0^9 - 6516L^2M^2r_0^{10} + 3740L^2Mr_0^{11} + 15L^2r_0^{12} + 480M^2r_0^{12} - 45r_0^{14}),
\end{aligned}$$

$$F_{t[6]} = \frac{-3E\dot{r}_0}{560\pi r_0^{16}(L^2 + r_0^2)^{15/2}}(F_{t[6]}^{\mathcal{E}}\mathcal{E} + F_{t[6]}^{\mathcal{K}}\mathcal{K}), \quad (5.38)$$

where

$$\begin{aligned}
F_{t[6]}^{\mathcal{E}} &= 28000E^6r_0^{23}(r_0 - L)(L + r_0)(11L^4 - 74L^2r_0^2 + 11r_0^4) \\
&\quad - 25E^4r_0^8(L^2 + r_0^2)(-16056320L^{18}M - 107151360L^{16}Mr_0^2 - 302586880L^{14}Mr_0^4 - 464979968L^{12}Mr_0^6 \\
&\quad - 412568652L^{10}Mr_0^8 - 201055024L^8Mr_0^{10} - 39268410L^6Mr_0^{12} - 1575L^6r_0^{13} + 5226426L^4Mr_0^{14} \\
&\quad + 99435L^4r_0^{15} + 3185118L^2Mr_0^{16} - 186165L^2r_0^{17} + 19662Mr_0^{18} + 35385r_0^{19}) \\
&\quad + 2E^2r_0^3(L^2 + r_0^2)^2(-1007616000L^{20}M^2 - 2885324800L^{18}M^2r_0^2 - 1548288000L^{18}Mr_0^3 \\
&\quad + 5271990272L^{16}M^2r_0^4 - 9940582400L^{16}Mr_0^5 + 35832487264L^{14}M^2r_0^6 - 27145052800L^{14}Mr_0^7 \\
&\quad + 69571689904L^{12}M^2r_0^8 - 40793731200L^{12}Mr_0^9 + 69887626312L^{10}M^2r_0^{10} - 36329433800L^{10}Mr_0^{11} \\
&\quad + 39015325900L^8M^2r_0^{12} - 19063343950L^8Mr_0^{13} + 11166709052L^6M^2r_0^{14} - 5373108900L^6Mr_0^{15} \\
&\quad + 7875L^6r_0^{16} + 1046817944L^4M^2r_0^{16} - 575985300L^4Mr_0^{17} + 94500L^4r_0^{18} - 118281276L^2M^2r_0^{18} \\
&\quad + 29440500L^2Mr_0^{19} - 1178625L^2r_0^{20} - 8271468M^2r_0^{20} + 479850Mr_0^{21} + 414750r_0^{22}) \\
&\quad + (L^2 + r_0^2)^3(-5775360000L^{20}M^3 + 2580480000L^{20}M^2r_0 - 18980904960L^{18}M^3r_0^2 + 750796800L^{18}M^2r_0^3 \\
&\quad + 3429888000L^{18}Mr_0^4 + 10876463104L^{16}M^3r_0^4 - 53063915520L^{16}M^2r_0^5 + 21396480000L^{16}Mr_0^6 \\
&\quad + 143196789568L^{14}M^3r_0^6 - 191859546624L^{14}M^2r_0^7 + 56793312800L^{14}Mr_0^8 + 292841560608L^{12}M^3r_0^8 \\
&\quad - 317782413664L^{12}M^2r_0^9 + 83096000800L^{12}Mr_0^{10} + 297880915104L^{10}M^3r_0^{10} - 295661821784L^{10}M^2r_0^{11} \\
&\quad + 72364880400L^{10}Mr_0^{12} + 168534399040L^8M^3r_0^{12} - 159472848000L^8M^2r_0^{13} + 37560515600L^8Mr_0^{14})
\end{aligned}$$

$$\begin{aligned}
& + 50707761864L^6M^3r_0^{14} - 46826640820L^6M^2r_0^{15} + 10839698800L^6Mr_0^{16} + 7000L^6r_0^{17} \\
& + 6272875728L^4M^3r_0^{16} - 5878415984L^4M^2r_0^{17} + 1398754050L^4Mr_0^{18} + 41125L^4r_0^{19} - 86931352L^2M^3r_0^{18} \\
& + 212436L^2M^2r_0^{19} + 21357300L^2Mr_0^{20} + 124250L^2r_0^{21} - 29620256M^3r_0^{20} + 16081176M^2r_0^{21} \\
& - 597150Mr_0^{22} - 245875r_0^{23}),
\end{aligned}$$

$$\begin{aligned}
F_{t[6]}^{\mathcal{K}} = & -875E^6r_0^{23}(-105L^6 + 1189L^4r_0^2 - 1531L^2r_0^4 + 247r_0^6) \\
& + 50E^4r_0^{10}(L^2 + r_0^2)(-4014080L^{16}M - 23275520L^{14}Mr_0^2 - 55468800L^{12}Mr_0^4 - 68718512L^{10}Mr_0^6 \\
& - 45183275L^8Mr_0^8 - 13052460L^6Mr_0^{10} + 362358L^4Mr_0^{12} + 18900L^4r_0^{13} + 919476L^2Mr_0^{14} - 49560L^2r_0^{15} \\
& + 15501Mr_0^{16} + 12180r_0^{17}) \\
& - E^2r_0^5(L^2 + r_0^2)^2(-1007616000L^{18}M^2 - 2003660800L^{16}M^2r_0^2 - 1548288000L^{16}Mr_0^3 \\
& + 6977961472L^{14}M^2r_0^4 - 8585830400L^{14}Mr_0^5 + 29653513376L^{12}M^2r_0^6 - 19705027200L^{12}Mr_0^7 \\
& + 43981240344L^{10}M^2r_0^8 - 23922541200L^{10}Mr_0^9 + 32635169200L^8M^2r_0^{10} - 16162950800L^8Mr_0^{11} \\
& + 12004692860L^6M^2r_0^{12} - 5726827800L^6Mr_0^{13} + 1578230992L^4M^2r_0^{14} - 803087700L^4Mr_0^{15} + 7875L^4r_0^{16} \\
& - 113069964L^2M^2r_0^{16} + 24092400L^2Mr_0^{17} - 1118250L^2r_0^{18} - 13324056M^2r_0^{18} + 1526700Mr_0^{19} + 553875r_0^{20}) \\
& + 2r_0^2(L^2 + r_0^2)^3(1443840000L^{18}M^3 - 645120000L^{18}M^2r_0 + 3481866240L^{16}M^3r_0^2 + 376780800L^{16}M^2r_0^3 \\
& - 857472000L^{16}Mr_0^4 - 5698068736L^{14}M^3r_0^4 + 12906055680L^{14}M^2r_0^5 - 4598832000L^{14}Mr_0^6 \\
& - 30679784768L^{12}M^3r_0^6 + 36702979536L^{12}M^2r_0^7 - 10214544200L^{12}Mr_0^8 - 46687160912L^{10}M^3r_0^8 \\
& + 47920180732L^{10}M^2r_0^9 - 12034259400L^{10}Mr_0^{10} - 34910457500L^8M^3r_0^{10} + 33450421620L^8M^2r_0^{11} \\
& - 7955786400L^8Mr_0^{12} - 13231827540L^6M^3r_0^{12} + 12237529680L^6M^2r_0^{13} - 2830778675L^6Mr_0^{14} \\
& - 2081061396L^4M^3r_0^{14} + 1920152080L^4M^2r_0^{15} - 448289925L^4Mr_0^{16} - 1750L^4r_0^{17} - 4481148L^2M^3r_0^{16} \\
& + 24248796L^2M^2r_0^{17} - 11278125L^2Mr_0^{18} - 8750L^2r_0^{19} + 11067088M^3r_0^{18} - 6431148M^2r_0^{19} \\
& + 440325Mr_0^{20} + 77000r_0^{21}),
\end{aligned}$$

$$F_{r[6]} = \frac{-3}{560\pi r_0^{18}(L^2 + r_0^2)^{15/2}(r_0 - 2M)} \left(F_{r[6]}^{\mathcal{E}} \mathcal{E} + F_{r[6]}^{\mathcal{K}} \mathcal{K} \right), \quad (5.39)$$

where

$$\begin{aligned}
F_{r[6]}^{\mathcal{E}} = & -28000E^8(r_0 - L)(L + r_0)(11L^4 - 74r_0^2L^2 + 11r_0^4)r_0^{26} \\
& + 50E^6(L^2 + r_0^2)(19565r_0^{19} + 6086Mr_0^{18} - 113785L^2r_0^{17} + 1633964L^2Mr_0^{16} + 76615L^4r_0^{15} + 2559418L^4Mr_0^{14} \\
& - 5075L^6r_0^{13} - 19625630L^6Mr_0^{12} - 100527512L^8Mr_0^{10} - 206284326L^{10}Mr_0^8 - 232489984L^{12}Mr_0^6 \\
& - 151293440L^{14}Mr_0^4 - 53575680L^{16}Mr_0^2 - 8028160L^{18}M)r_0^{11} \\
& - E^4(L^2 + r_0^2)^2(1094625r_0^{22} + 545450Mr_0^{21} - 4202625L^2r_0^{20} - 16774936M^2r_0^{20} + 110101000L^2Mr_0^{19} \\
& + 1414875L^4r_0^{18} - 331621052L^2M^2r_0^{18} - 822447000L^4Mr_0^{17} - 7875L^6r_0^{16} + 1429685188L^4M^2r_0^{16} \\
& - 9480504900L^6Mr_0^{15} + 19802086804L^6M^2r_0^{14} - 35145688050L^8Mr_0^{13} + 72068652100L^8M^2r_0^{12} \\
& - 68030273700L^{10}Mr_0^{11} + 130518064824L^{10}M^2r_0^{10} - 76873222400L^{12}Mr_0^9 + 129714899808L^{12}M^2r_0^8 \\
& - 51274604800L^{14}Mr_0^7 + 65633972928L^{14}M^2r_0^6 - 18785894400L^{16}Mr_0^5 + 8353439744L^{16}M^2r_0^4 \\
& - 2924544000L^{18}Mr_0^3 - 6114713600L^{18}M^2r_0^2 - 2015232000L^{20}M^2)r_0^6 \\
& + 2E^2(L^2 + r_0^2)^3(242375r_0^{23} + 138500Mr_0^{22} - 419125L^2r_0^{21} - 10992820M^2r_0^{21} + 20399392M^3r_0^{20} \\
& + 6959650L^2Mr_0^{20} + 9625L^4r_0^{19} - 81665756L^2M^2r_0^{19} + 138887552L^2M^3r_0^{18} - 822884550L^4Mr_0^{18} \\
& - 875L^6r_0^{17} + 3518901164L^4M^2r_0^{17} - 3802035608L^4M^3r_0^{16} - 7067551050L^6Mr_0^{16} + 30648837404L^6M^2r_0^{15} \\
& - 33234129320L^6M^3r_0^{14} - 25352340750L^8Mr_0^{14} + 106617027800L^8M^2r_0^{13} - 111740075320L^8M^3r_0^{12} \\
& - 49660036200L^{10}Mr_0^{12} + 197830827680L^{10}M^2r_0^{11} - 195029907128L^{10}M^3r_0^{10} - 57553509200L^{12}Mr_0^{10} \\
& + 209550665904L^{12}M^2r_0^9 - 183717663248L^{12}M^3r_0^8 - 39556896400L^{14}Mr_0^8 + 121346652064L^{14}M^2r_0^7 \\
& - 77791192288L^{14}M^3r_0^6 - 14956032000L^{16}Mr_0^6 + 28755231744L^{16}M^2r_0^5 + 7146388480L^{16}M^3r_0^4 \\
& - 2403072000L^{18}Mr_0^4 - 3619737600L^{18}M^2r_0^3 + 18731642880L^{18}M^3r_0^2 - 2297856000L^{20}M^2r_0
\end{aligned}$$

$$\begin{aligned}
& + 4902912000L^{20}M^3)r_0^3 \\
& + (2M - r_0)(L^2 + r_0^2)^4(53375r_0^{23} + 173600Mr_0^{22} + 29750L^2r_0^{21} - 5212944M^2r_0^{21} + 10317184M^3r_0^{20} \\
& \quad - 14183750L^2Mr_0^{20} + 25375L^4r_0^{19} + 62499436L^2M^2r_0^{19} - 76153328L^2M^3r_0^{18} - 679641900L^4Mr_0^{18} \\
& \quad + 7000L^6r_0^{17} + 3345821696L^4M^2r_0^{17} - 4111469784L^4M^3r_0^{16} - 5518261350L^6Mr_0^{16} + 25823273900L^6M^2r_0^{15} \\
& \quad - 30032966752L^6M^3r_0^{14} - 20150264400L^8Mr_0^{14} + 88797134680L^8M^2r_0^{13} - 96877958296L^8M^3r_0^{12} \\
& \quad - 40738068000L^{10}Mr_0^{12} + 166165277616L^{10}M^2r_0^{11} - 165473393280L^{10}M^3r_0^{10} - 48857572400L^{12}Mr_0^{10} \\
& \quad + 177451524416L^{12}M^2r_0^9 - 149256330784L^{12}M^3r_0^8 - 34733020000L^{14}Mr_0^8 + 101711045376L^{14}M^2r_0^7 \\
& \quad - 51263318208L^{14}M^3r_0^6 - 13563648000L^{16}Mr_0^6 + 21232814080L^{16}M^2r_0^5 + 21391316992L^{16}M^3r_0^4 \\
& \quad - 2247168000L^{18}Mr_0^4 - 5520998400L^{18}M^2r_0^3 + 23777402880L^{18}M^3r_0^2 - 2580480000L^{20}M^2r_0 \\
& \quad + 5775360000L^{20}M^3),
\end{aligned}$$

$$\begin{aligned}
F_{r[6]}^{\mathcal{K}} = & 875E^8(-105L^6 + 1189r_0^2L^4 - 1531r_0^4L^2 + 247r_0^6)r_0^{26} \\
& - 25E^6(L^2 + r_0^2)(27055r_0^{17} + 25612Mr_0^{16} - 123235L^2r_0^{15} + 1887182L^2Mr_0^{14} + 62125L^4r_0^{13} + 676066L^4Mr_0^{12} \\
& \quad - 2625L^6r_0^{11} - 26099670L^6Mr_0^{10} - 90366550L^8Mr_0^8 - 137437024L^{10}Mr_0^6 - 110937600L^{12}Mr_0^4 \\
& \quad - 46551040L^{14}Mr_0^2 - 8028160L^{16}M)r_0^{13} \\
& + 2E^4(L^2 + r_0^2)^2(370125r_0^{20} + 697975Mr_0^{19} - 1065750L^2r_0^{18} - 6904028M^2r_0^{18} + 27977700L^2Mr_0^{17} \\
& \quad + 244125L^4r_0^{16} - 86371482L^2M^2r_0^{16} - 315079050L^4Mr_0^{15} + 615225146L^4M^2r_0^{14} - 2582807100L^6Mr_0^{13} \\
& \quad + 5441132830L^6M^2r_0^{12} - 7522573725L^8Mr_0^{11} + 15199781250L^8M^2r_0^{10} - 11253096600L^{10}Mr_0^9 \\
& \quad + 20574272172L^{10}M^2r_0^8 - 9303284800L^{12}Mr_0^7 + 13728299088L^{12}M^2r_0^6 - 4056729600L^{14}Mr_0^5 \\
& \quad + 3016609536L^{14}M^2r_0^4 - 731136000L^{16}Mr_0^3 - 1087846400L^{16}M^2r_0^2 - 503808000L^{18}M^2)r_0^8 \\
& - E^2(L^2 + r_0^2)^3(314125r_0^{21} + 970000Mr_0^{20} - 358750L^2r_0^{19} - 18222440M^2r_0^{19} + 31216064M^3r_0^{18} \\
& \quad - 3780450L^2Mr_0^{18} - 875L^4r_0^{17} - 36108732L^2M^2r_0^{17} + 87877152L^2M^3r_0^{16} - 1092352500L^4Mr_0^{16} \\
& \quad + 4756636984L^4M^2r_0^{15} - 5211718840L^4M^3r_0^{14} - 7494234450L^6Mr_0^{14} + 32418083300L^6M^2r_0^{13} \\
& \quad - 35018994560L^6M^3r_0^{12} - 21674793600L^8Mr_0^{12} + 89821855320L^8M^2r_0^{11} - 92610055960L^8M^3r_0^{10} \\
& \quad - 33236721600L^{10}Mr_0^{10} + 127823271040L^{10}M^2r_0^9 - 120667329776L^{10}M^3r_0^8 - 28422864400L^{12}Mr_0^8 \\
& \quad + 95019791488L^{12}M^2r_0^7 - 72612130368L^{12}M^3r_0^6 - 12853344000L^{14}Mr_0^6 + 30055494144L^{14}M^2r_0^5 \\
& \quad - 5260183040L^{14}M^3r_0^4 - 2403072000L^{16}Mr_0^4 - 1609113600L^{16}M^2r_0^3 + 14441594880L^{16}M^3r_0^2 \\
& \quad - 2297856000L^{18}M^2r_0 + 4902912000L^{18}M^3)r_0^5 \\
& + (r_0 - 2M)(L^2 + r_0^2)^4(27125r_0^{21} + 299600Mr_0^{20} + 9625L^2r_0^{19} - 4164624M^2r_0^{19} + 7521664M^3r_0^{18} \\
& \quad - 12312300L^2Mr_0^{18} + 3500L^4r_0^{17} + 60136468L^2M^2r_0^{17} - 77649520L^2M^3r_0^{16} - 435265950L^4Mr_0^{16} \\
& \quad + 2136130980L^4M^2r_0^{15} - 2610250432L^4M^3r_0^{14} - 2918350050L^6Mr_0^{14} + 13485739400L^6M^2r_0^{13} \\
& \quad - 15471992072L^6M^3r_0^{12} - 8700997200L^8Mr_0^{12} + 37444962000L^8M^2r_0^{11} - 39767055768L^8M^3r_0^{10} \\
& \quad - 13875397200L^{10}Mr_0^{10} + 54025898376L^{10}M^2r_0^9 - 50546368608L^{10}M^3r_0^8 - 12345326000L^{12}Mr_0^8 \\
& \quad + 40319920928L^{12}M^2r_0^7 - 27561410368L^{12}M^3r_0^6 - 5798688000L^{14}Mr_0^6 + 11983523840L^{14}M^2r_0^5 \\
& \quad + 2639284736L^{14}M^3r_0^4 - 1123584000L^{16}Mr_0^4 - 1631539200L^{16}M^2r_0^3 + 9361981440L^{16}M^3r_0^2 \\
& \quad - 1290240000L^{18}M^2r_0 + 2887680000L^{18}M^3)r_0^2,
\end{aligned}$$

$$F_{\theta[6]} = 0, \quad (5.40)$$

$$F_{\phi[6]} = \frac{-3\dot{r}_0}{560\pi Lr_0^{16}(L^2 + r_0^2)^{13/2}} \left(F_{\phi[6]}^{\mathcal{E}} \mathcal{E} + F_{\phi[6]}^{\mathcal{K}} \mathcal{K} \right), \quad (5.41)$$

where

$$\begin{aligned}
F_{\phi[6]}^{\mathcal{E}} = & 875E^6r_0^{23} (1773L^4r_0^2 - 337L^6 - 947L^2r_0^4 + 15r_0^6) \\
& + 175E^4r_0^8 (L^2 + r_0^2) (983040L^{18}M + 7045120L^{16}Mr_0^2 + 21570560L^{14}Mr_0^4 + 36582912L^{12}Mr_0^6 \\
& + 37139572L^{10}Mr_0^8 + 22617566L^8Mr_0^{10} + 7708830L^6Mr_0^{12} + 225L^6r_0^{13} + 1199730L^4Mr_0^{14} - 9375L^4r_0^{15} \\
& + 4926L^2Mr_0^{16} + 9375L^2r_0^{17} - 225r_0^{19}) \\
& + E^2r_0^3 (L^2 + r_0^2)^2 (2015232000L^{20}M^2 + 8400691200L^{18}M^2r_0^2 + 1376256000L^{18}Mr_0^3 + 11063078912L^{16}M^2r_0^4 \\
& + 7276953600L^{16}Mr_0^5 + 82626752L^{14}M^2r_0^6 + 15631481600L^{14}Mr_0^7 - 12481032128L^{12}M^2r_0^8 \\
& + 17141443200L^{12}Mr_0^9 - 10632497080L^{10}M^2r_0^{10} + 9613116800L^{10}Mr_0^{11} - 2120762600L^8M^2r_0^{12} \\
& + 2046335900L^8Mr_0^{13} + 1035946292L^6M^2r_0^{14} - 316454600L^6Mr_0^{15} + 15750L^6r_0^{16} + 431941952L^4M^2r_0^{16} \\
& - 166655300L^4Mr_0^{17} + 133875L^4r_0^{18} + 17346492L^2M^2r_0^{18} - 2290400L^2Mr_0^{19} - 850500L^2r_0^{20} \\
& - 312480M^2r_0^{20} + 39375r_0^{22}) \\
& - (L^2 + r_0^2)^3 (2580480000L^{20}M^2r_0 - 5775360000L^{20}M^3 - 21381242880L^{18}M^3r_0^2 + 4448665600L^{18}M^2r_0^3 \\
& + 2247168000L^{18}Mr_0^4 - 16436058112L^{16}M^3r_0^4 - 20228515840L^{16}M^2r_0^5 + 12144384000L^{16}Mr_0^6 \\
& + 37967372736L^{14}M^3r_0^6 - 78865174016L^{14}M^2r_0^7 + 27288335200L^{14}Mr_0^8 + 91117248928L^{12}M^3r_0^8 \\
& - 114866020480L^{12}M^2r_0^9 + 32738062000L^{12}Mr_0^{10} + 80974789248L^{10}M^3r_0^{10} - 86565871136L^{10}M^2r_0^{11} \\
& + 22304895200L^{10}Mr_0^{12} + 35112838392L^8M^3r_0^{12} - 34540540744L^8M^2r_0^{13} + 8392988800L^8Mr_0^{14} \\
& + 6757023136L^6M^3r_0^{14} - 6373891596L^6M^2r_0^{15} + 1516716950L^6Mr_0^{16} - 7000L^6r_0^{17} + 273916728L^4M^3r_0^{16} \\
& - 286552800L^4M^2r_0^{17} + 80573500L^4Mr_0^{18} - 28875L^4r_0^{19} - 24711184L^2M^3r_0^{18} + 14372004L^2M^2r_0^{19} \\
& - 855050L^2Mr_0^{20} - 50750L^2r_0^{21} + 309120M^3r_0^{20} - 245280M^2r_0^{21} + 13125r_0^{23}),
\end{aligned}$$

$$\begin{aligned}
F_{\phi[6]}^{\mathcal{K}} = & -875E^6r_0^{23} (-105L^6 + 829L^4r_0^2 - 587L^2r_0^4 + 15r_0^6) \\
& + 175E^4r_0^{10} (L^2 + r_0^2) (-491520L^{16}M - 3092480L^{14}Mr_0^2 - 8102400L^{12}Mr_0^4 - 11336736L^{10}Mr_0^6 \\
& - 8972282L^8Mr_0^8 - 3855372L^6Mr_0^{10} - 750282L^4Mr_0^{12} + 3825L^4r_0^{13} - 12696L^2Mr_0^{14} \\
& - 5550L^2r_0^{15} + 225r_0^{17}) \\
& - E^2r_0^5 (L^2 + r_0^2)^2 (1007616000L^{18}M^2 + 3318681600L^{16}M^2r_0^2 + 688128000L^{16}Mr_0^3 + 2674925056L^{14}M^2r_0^4 \\
& + 3036364800L^{14}Mr_0^5 - 2164346848L^{12}M^2r_0^6 + 5191177600L^{12}Mr_0^7 - 4277576360L^{10}M^2r_0^8 \\
& + 4156658800L^{10}Mr_0^9 - 1698491920L^8M^2r_0^{10} + 1358613200L^8Mr_0^{11} + 281096500L^6M^2r_0^{12} \\
& - 46924500L^6Mr_0^{13} + 248366216L^4M^2r_0^{14} - 97522600L^4Mr_0^{15} + 7875L^4r_0^{16} + 15819372L^2M^2r_0^{16} \\
& - 3686900L^2Mr_0^{17} - 456750L^2r_0^{18} - 312480M^2r_0^{18} + 39375r_0^{20}) \\
& + r_0^2 (L^2 + r_0^2)^3 (-2887680000L^{18}M^3 + 1290240000L^{18}M^2r_0 - 8163901440L^{16}M^3r_0^2 + 1095372800L^{16}M^2r_0^3 \\
& + 1123584000L^{16}Mr_0^4 - 1209975296L^{14}M^3r_0^4 - 11012229120L^{14}M^2r_0^5 + 5089056000L^{14}Mr_0^6 \\
& + 19718951872L^{12}M^3r_0^6 - 29772000928L^{12}M^2r_0^7 + 9243911600L^{12}Mr_0^8 + 28381407744L^{10}M^3r_0^8 \\
& - 31906051368L^{10}M^2r_0^9 + 8496115600L^{10}Mr_0^{10} + 16529207256L^8M^3r_0^{10} - 16531893472L^8M^2r_0^{11} \\
& + 4060456400L^8Mr_0^{12} + 4051974744L^6M^3r_0^{12} - 3820351368L^6M^2r_0^{13} + 903318850L^6Mr_0^{14} \\
& + 233659760L^4M^3r_0^{14} - 228732252L^4M^2r_0^{15} + 59520650L^4Mr_0^{16} - 3500L^4r_0^{17} - 17332624L^2M^3r_0^{16} \\
& + 10640724L^2M^2r_0^{17} - 891800L^2Mr_0^{18} - 11375L^2r_0^{19} + 309120M^3r_0^{18} - 245280M^2r_0^{19} + 13125r_0^{21}).
\end{aligned}$$

D. Electromagnetic case

In the electromagnetic case, an ambiguity arises in the definition of u^a in the angular directions *away* from the world-line. In (2.19) one is free to define $u^a(x)$ as they wish provided $\lim_{x \rightarrow \bar{x}} u^a(x) = u^{\bar{a}}$. A natural covariant choice would be to define this through parallel transport, $u^a(x) = g^a{}_{\bar{b}} u^{\bar{b}}$. However, in reality it is more practical in numerical calculations to define u^a such that its components in Schwarzschild coordinates are equal to the components of $u^{\bar{a}}$ in Schwarzschild coordinates. Doing so, the regularization parameters are given by

$$F_{t[-1]} = -\frac{\dot{r}_0 \operatorname{sgn}(\Delta r)}{2(L^2 + r_0^2)}, \quad F_{r[-1]} = \frac{Er_0 \operatorname{sgn}(\Delta r)}{2(r_0 - 2M)(L^2 + r_0^2)}, \quad F_{\theta[-1]} = 0, \quad F_{\phi[-1]} = 0, \quad (5.42)$$

$$F_{t[0]} = -\frac{E\dot{r}_0}{\pi r_0 (r_0^2 + L^2)^{3/2}} (r_0^2 \mathcal{K} + 2L^2 \mathcal{E}), \quad (5.43)$$

$$F_{r[0]} = \frac{1}{\pi r_0^3 (r_0^2 + L^2)^{3/2} (r_0 - 2M)} (F_{r[0]}^{\mathcal{E}} \mathcal{E} + F_{r[0]}^{\mathcal{K}} \mathcal{K}), \quad (5.44)$$

where

$$\begin{aligned} F_{r[0]}^{\mathcal{E}} &= 2E^2 L^2 r_0^3 + (L^2 + r_0^2) (2L^2 + r_0^2) (2M - r_0), \\ F_{r[0]}^{\mathcal{K}} &= E^2 r_0^5 + r_0^2 (L^2 + r_0^2) (r_0 - 2M), \end{aligned}$$

$$F_{\theta[0]} = 0, \quad (5.45)$$

$$F_{\phi[0]} = \frac{\dot{r}_0}{\pi L r_0 \sqrt{L^2 + r_0^2}} [\mathcal{E} (2L^2 + r_0^2) - \mathcal{K} r_0^2], \quad (5.46)$$

$$F_{t[2]} = -\frac{E\dot{r}_0}{2\pi r_0^4 (L^2 + r_0^2)^{7/2}} (F_{t[2]}^{\mathcal{E}} \mathcal{E} + F_{t[2]}^{\mathcal{K}} \mathcal{K}), \quad (5.47)$$

where

$$\begin{aligned} F_{t[2]}^{\mathcal{E}} &= 2E^2 r_0^5 (-L^4 + 10L^2 r_0^2 + 3r_0^4) + (L^2 + r_0^2) (60L^6 M + 168L^4 M r_0^2 + 182L^2 M r_0^4 - 13L^2 r_0^5 + 58M r_0^6 - 5r_0^7), \\ F_{t[2]}^{\mathcal{K}} &= 2r_0^2 (L^2 + r_0^2) (-21L^4 M - 48L^2 M r_0^2 + 3L^2 r_0^3 - 23M r_0^4 + r_0^5) - E^2 r_0^7 (11L^2 + 3r_0^2), \end{aligned}$$

$$F_{r[2]} = -\frac{1}{2\pi r_0^6 (L^2 + r_0^2)^{7/2} (r_0 - 2M)} (F_{r[2]}^{\mathcal{E}} \mathcal{E} + F_{r[2]}^{\mathcal{K}} \mathcal{K}), \quad (5.48)$$

where

$$\begin{aligned} F_{r[2]}^{\mathcal{E}} &= -2E^4 r_0^8 (-L^4 + 10L^2 r_0^2 + 3r_0^4) - (L^2 + r_0^2)^2 (2M - r_0) (44L^6 M + 94L^4 M r_0^2 + 54L^2 M r_0^4 + L^2 r_0^5 + 3r_0^7) \\ &\quad + 2E^2 r_0^3 (L^2 + r_0^2) (-30L^6 M - 86L^4 M r_0^2 + L^4 r_0^3 - 98L^2 M r_0^4 + 10L^2 r_0^5 - 26M r_0^6 + r_0^7), \\ F_{r[2]}^{\mathcal{K}} &= E^4 r_0^{10} (11L^2 + 3r_0^2) - r_0^2 (L^2 + r_0^2)^2 (r_0 - 2M) (22L^4 M + 24L^2 M r_0^2 + 2L^2 r_0^3 + 3r_0^5) \\ &\quad + E^2 r_0^5 (L^2 + r_0^2) (42L^4 M + 98L^2 M r_0^2 - 7L^2 r_0^3 + 40M r_0^4 + r_0^5), \end{aligned}$$

$$F_{\theta[2]} = 0, \quad (5.49)$$

$$F_{\phi[2]} = -\frac{\dot{r}_0}{2\pi L r_0^4 (L^2 + r_0^2)^{5/2}} (F_{\phi[2]}^{\mathcal{E}} \mathcal{E} + F_{\phi[2]}^{\mathcal{K}} \mathcal{K}), \quad (5.50)$$

where

$$\begin{aligned} F_{\phi[2]}^{\mathcal{E}} &= E^2 r_0^5 (-2L^4 - 7L^2 r_0^2 + 3r_0^4) - (L^2 + r_0^2) (44L^6 M + 94L^4 M r_0^2 + 54L^2 M r_0^4 + L^2 r_0^5 + 3r_0^7), \\ F_{\phi[2]}^{\mathcal{K}} &= r_0^2 (L^2 + r_0^2) (22L^4 M + 24L^2 M r_0^2 + 2L^2 r_0^3 + 3r_0^5) - E^2 r_0^7 (3r_0^2 - L^2), \end{aligned}$$

$$F_{t[4]} = \frac{3E\dot{r}_0}{40\pi r_0^{11} (L^2 + r_0^2)^{11/2}} (F_{t[4]}^{\mathcal{E}} \mathcal{E} + F_{t[4]}^{\mathcal{K}} \mathcal{K}), \quad (5.51)$$

where

$$\begin{aligned}
F_{t[4]}^{\mathcal{E}} &= -30E^4r_0^{14} (3L^6 - 102L^4r_0^2 + 43L^2r_0^4 + 20r_0^6) \\
&\quad + 2E^2r_0^5 (L^2 + r_0^2) (34560L^{12}M + 169728L^{10}Mr_0^2 + 333564L^8Mr_0^4 + 328912L^6Mr_0^6 + 167074L^4Mr_0^8 \\
&\quad \quad - 1245L^4r_0^9 + 32948L^2Mr_0^{10} + 1230L^2r_0^{11} + 230Mr_0^{12} + 555r_0^{13}) \\
&\quad - 4(L^2 + r_0^2)^2 (11520L^{14}M^2 - 18240L^{12}M^2r_0^2 + 33600L^{12}Mr_0^3 - 226624L^{10}M^2r_0^4 + 153960L^{10}Mr_0^5 \\
&\quad \quad - 496164L^8M^2r_0^6 + 280764L^8Mr_0^7 - 485652L^6M^2r_0^8 + 255197L^6Mr_0^9 - 230930L^4M^2r_0^{10} + 116771L^4Mr_0^{11} \\
&\quad \quad - 30L^4r_0^{12} - 45200L^2M^2r_0^{12} + 21487L^2Mr_0^{13} + 270L^2r_0^{14} - 1342M^2r_0^{14} + 229Mr_0^{15} + 120r_0^{16}), \\
F_{t[4]}^{\mathcal{K}} &= 15E^4r_0^{16} (-87L^4 + 66L^2r_0^2 + 25r_0^4) \\
&\quad - 4E^2r_0^7 (L^2 + r_0^2) (8640L^{10}M + 34872L^8Mr_0^2 + 53283L^6Mr_0^4 + 37555L^4Mr_0^6 - 225L^4r_0^7 + 9809L^2Mr_0^8 \\
&\quad \quad + 420L^2r_0^9 + 265Mr_0^{10} + 165r_0^{11}) \\
&\quad + r_0^2 (L^2 + r_0^2)^2 (23040L^{12}M^2 - 56640L^{10}M^2r_0^2 + 67200L^{10}Mr_0^3 - 402608L^8M^2r_0^4 + 249120L^8Mr_0^5 \\
&\quad \quad - 643736L^6M^2r_0^6 + 346608L^6Mr_0^7 - 427796L^4M^2r_0^8 + 217192L^4Mr_0^9 - 110916L^2M^2r_0^{10} \\
&\quad \quad + 52580L^2Mr_0^{11} + 615L^2r_0^{12} - 5368M^2r_0^{12} + 1516Mr_0^{13} + 255r_0^{14}),
\end{aligned}$$

$$F_{r[4]} = \frac{3}{40\pi r_0^{13} (L^2 + r_0^2)^{11/2} (r_0 - 2M)} (F_{r[4]}^{\mathcal{E}} \mathcal{E} + F_{r[4]}^{\mathcal{K}} \mathcal{K}), \quad (5.52)$$

where

$$\begin{aligned}
F_{r[4]}^{\mathcal{E}} &= 30E^6r_0^{17} (3L^6 - 102L^4r_0^2 + 43L^2r_0^4 + 20r_0^6) \\
&\quad - E^4r_0^8 (L^2 + r_0^2) (69120L^{12}M + 339456L^{10}Mr_0^2 + 667128L^8Mr_0^4 + 657884L^6Mr_0^6 - 30L^6r_0^7 + 334658L^4Mr_0^8 \\
&\quad \quad - 2745L^4r_0^9 + 62656L^2Mr_0^{10} + 4080L^2r_0^{11} + 610Mr_0^{12} + 1035r_0^{13}) \\
&\quad + (L^2 + r_0^2)^3 (2M - r_0) (46080L^{14}M^2 + 89856L^{12}M^2r_0^2 + 53760L^{12}Mr_0^3 - 86336L^{10}M^2r_0^4 + 211520L^{10}Mr_0^5 \\
&\quad \quad - 344128L^8M^2r_0^6 + 317600L^8Mr_0^7 - 306808L^6M^2r_0^8 + 221140L^6Mr_0^9 - 98676L^4M^2r_0^{10} + 66220L^4Mr_0^{11} \\
&\quad \quad + 60L^4r_0^{12} - 5244L^2M^2r_0^{12} + 4440L^2Mr_0^{13} + 105L^2r_0^{14} + 160M^2r_0^{14} - 75r_0^{16}) \\
&\quad + 2E^2r_0^3 (L^2 + r_0^2)^2 (23040L^{14}M^2 - 36480L^{12}M^2r_0^2 + 67200L^{12}Mr_0^3 - 445056L^{10}M^2r_0^4 + 303824L^{10}Mr_0^5 \\
&\quad \quad - 959952L^8M^2r_0^6 + 545340L^8Mr_0^7 - 922996L^6M^2r_0^8 + 486240L^6Mr_0^9 - 428644L^4M^2r_0^{10} + 216604L^4Mr_0^{11} \\
&\quad \quad + 105L^4r_0^{12} - 78428L^2M^2r_0^{12} + 35368L^2Mr_0^{13} + 1350L^2r_0^{14} - 2684M^2r_0^{14} + 608Mr_0^{15} + 165r_0^{16}), \\
F_{r[4]}^{\mathcal{K}} &= -15E^6r_0^{19} (-87L^4 + 66L^2r_0^2 + 25r_0^4) \\
&\quad + E^4r_0^{10} (L^2 + r_0^2) (34560L^{10}M + 139488L^8Mr_0^2 + 213132L^6Mr_0^4 + 150250L^4Mr_0^6 - 915L^4r_0^7 + 37496L^2Mr_0^8 \\
&\quad \quad + 2550L^2r_0^9 + 1210Mr_0^{10} + 585r_0^{11}) \\
&\quad - r_0^2 (L^2 + r_0^2)^3 (r_0 - 2M) (63760L^8M^2r_0^4 - 23040L^{12}M^2 - 24768L^{10}M^2r_0^2 - 26880L^{10}Mr_0^3 - 82240L^8Mr_0^5 \\
&\quad \quad + 115848L^6M^2r_0^6 - 87920L^6Mr_0^7 + 56084L^4M^2r_0^8 - 36340L^4Mr_0^9 + 5324L^2M^2r_0^{10} - 3540L^2Mr_0^{11} \\
&\quad \quad + 15L^2r_0^{12} - 160M^2r_0^{12} + 75r_0^{14}) \\
&\quad - E^2r_0^5 (L^2 + r_0^2)^2 (23040L^{12}M^2 - 56640L^{10}M^2r_0^2 + 67200L^{10}Mr_0^3 - 394416L^8M^2r_0^4 + 245024L^8Mr_0^5 \\
&\quad \quad - 618528L^6M^2r_0^6 + 334004L^6Mr_0^7 - 400636L^4M^2r_0^8 + 203252L^4Mr_0^9 + 180L^4r_0^{10} - 97892L^2M^2r_0^{10} \\
&\quad \quad + 44568L^2Mr_0^{11} + 1365L^2r_0^{12} - 5368M^2r_0^{12} + 1816Mr_0^{13} + 105r_0^{14}),
\end{aligned}$$

$$F_{\theta[4]} = 0, \quad (5.53)$$

$$F_{\phi[4]} = \frac{3\dot{r}_0}{40\pi Lr_0^{11} (L^2 + r_0^2)^{9/2}} (F_{\phi[4]}^{\mathcal{E}} \mathcal{E} + F_{\phi[4]}^{\mathcal{K}} \mathcal{K}), \quad (5.54)$$

where

$$\begin{aligned}
F_{\phi[4]}^{\mathcal{E}} &= -15E^4r_0^{14}(2L^6 + 17L^4r_0^2 - 108L^2r_0^4 + 5r_0^6) \\
&\quad + 2E^2r_0^7(L^2 + r_0^2)(4096L^{10}M + 16188L^8Mr_0^2 + 24154L^6Mr_0^4 + 16608L^4Mr_0^6 - 165L^4r_0^7 + 5986L^2Mr_0^8 \\
&\quad \quad - 810L^2r_0^9 + 75r_0^{11}) \\
&\quad + (L^2 + r_0^2)^2(46080L^{14}M^2 + 89856L^{12}M^2r_0^2 + 53760L^{12}Mr_0^3 - 86336L^{10}M^2r_0^4 + 211520L^{10}Mr_0^5 \\
&\quad \quad - 344128L^8M^2r_0^6 + 317600L^8Mr_0^7 - 306808L^6M^2r_0^8 + 221140L^6Mr_0^9 - 98676L^4M^2r_0^{10} \\
&\quad \quad + 66220L^4Mr_0^{11} + 60L^4r_0^{12} - 5244L^2M^2r_0^{12} + 4440L^2Mr_0^{13} + 105L^2r_0^{14} + 160M^2r_0^{14} - 75r_0^{16}), \\
F_{\phi[4]}^{\mathcal{K}} &= 15E^4r_0^{16}(L^4 - 58L^2r_0^2 + 5r_0^4) \\
&\quad - 2E^2r_0^9(L^2 + r_0^2)(2048L^8M + 6302L^6Mr_0^2 + 6790L^4Mr_0^4 - 90L^4r_0^5 + 3256L^2Mr_0^6 - 375L^2r_0^7 + 75r_0^9) \\
&\quad + r_0^2(L^2 + r_0^2)^2(-23040L^{12}M^2 - 24768L^{10}M^2r_0^2 - 26880L^{10}Mr_0^3 + 63760L^8M^2r_0^4 - 82240L^8Mr_0^5 \\
&\quad \quad + 115848L^6M^2r_0^6 - 87920L^6Mr_0^7 + 56084L^4M^2r_0^8 - 36340L^4Mr_0^9 + 5324L^2M^2r_0^{10} - 3540L^2Mr_0^{11} \\
&\quad \quad + 15L^2r_0^{12} - 160M^2r_0^{12} + 75r_0^{14}).
\end{aligned}$$

E. Gravitational case

1. Self-force regularization

The self force on a gravitational particle is given by

$$F^a = k^{abcd}\bar{h}_{bc;d}^R, \quad (5.55)$$

where

$$k^{abcd} \equiv \frac{1}{2}g^{ad}u^bu^c - g^{ab}u^cu^d - \frac{1}{2}u^au^bu^cu^d + \frac{1}{4}u^a g^{bc}u^d + \frac{1}{4}g^{ad}g^{bc}. \quad (5.56)$$

Note that, as in the electromagnetic case, an ambiguity arises here due to the presence of terms involving the four-velocity at x . One is free to arbitrarily choose how to define this provided $\lim_{x \rightarrow \bar{x}} u^a = u^{\bar{a}}$. Following Barack and Sago [63], we choose to take the Schwarzschild components of the four velocity at x to be exactly those at \bar{x} . The regularisation parameters in the gravitational case are given by

$$F_{[-1]}^t = \mp \frac{r_0 \dot{r}_0}{2(L^2 + r_0^2)(r_0 - 2M)}, \quad F_{[-1]}^r = \mp \frac{E}{2(L^2 + r_0^2)}, \quad F_{[-1]}^\theta = 0, \quad F_{[-1]}^\phi = 0, \quad (5.57)$$

$$F_{[0]}^t = -\frac{\dot{r}_0 E}{\pi(r_0 - 2M)(L^2 + r_0^2)^{3/2}}(2L^2\mathcal{E} + r_0^2\mathcal{K}), \quad (5.58)$$

$$\begin{aligned}
F_{[0]}^r &= \frac{1}{\pi r_0^4(L^2 + r_0^2)^{3/2}} \left\{ [(r_0 - 2M)(L^2 + r_0^2)(2L^2 + r_0^2) - 2E^2L^2r_0^3]\mathcal{E} - r_0^2[(r_0 - 2M)(L^2 + r_0^2) + E^2r_0^3]\mathcal{K} \right\}, \\
F_{[0]}^\theta &= 0, \quad F_{[0]}^\phi = -\frac{\dot{r}_0}{\pi Lr_0^3(L^2 + r_0^2)^{1/2}}[(2L^2 + r_0^2)\mathcal{E} - r_0^2\mathcal{K}], \quad (5.59)
\end{aligned}$$

$$F_{[2]}^t = \frac{E\dot{r}_0}{2\pi r_0^3(r_0 - 2M)(L^2 + r_0^2)^{7/2}}(F_{\mathcal{E}[2]}^t\mathcal{E} + D_{\mathcal{K}[2]}^t\mathcal{K}), \quad (5.60)$$

where

$$\begin{aligned}
F_{\mathcal{E}[2]}^t &= -2E^2r_0^5(11L^4 + 34L^2r_0^2 + 15r_0^4) \\
&\quad - (L^2 + r_0^2)(276L^6M + 768L^4Mr_0^2 + 782L^2Mr_0^4 - 37L^2r_0^5 + 274Mr_0^6 - 29r_0^7), \\
F_{\mathcal{K}[2]}^t &= E^2r_0^5(12L^4 + 35L^2r_0^2 + 15r_0^4) + 2r_0^2(L^2 + r_0^2)(93L^4M + 204L^2Mr_0^2 - 9L^2r_0^3 + 107Mr_0^4 - 7r_0^5),
\end{aligned}$$

$$F_{[2]}^r = \frac{1}{2\pi r_0^7(L^2 + r_0^2)^{7/2}}(F_{\mathcal{E}[2]}^r \mathcal{E} + F_{\mathcal{K}[2]}^r \mathcal{K}), \quad (5.61)$$

where

$$\begin{aligned} F_{\mathcal{E}[2]}^r &= (r_0 - 2M)(L^2 + r_0^2)^2(188L^6M + 406L^4Mr_0^2 + 222L^2Mr_0^4 + 13L^2r_0^5 + 15r_0^7) - 2E^4r_0^8(11L^4 + 34L^2r_0^2 + 15r_0^4) \\ &\quad - 2E^2r_0^3(L^2 + r_0^2)(138L^6M + 422L^4Mr_0^2 - 19L^4r_0^3 + 422L^2Mr_0^4 - 34L^2r_0^5 + 122Mr_0^6 - 7r_0^7), \\ F_{\mathcal{K}[2]}^r &= -r_0^2(r_0 - 2M)(L^2 + r_0^2)^2(94L^4M + 96L^2Mr_0^2 + 14L^2r_0^3 + 15r_0^5) + E^4r_0^8(12L^4 + 35L^2r_0^2 + 15r_0^4) \\ &\quad + E^2r_0^5(L^2 + r_0^2)(210L^4M - 12L^4r_0 + 410L^2Mr_0^2 - 19L^2r_0^3 + 184Mr_0^4 + r_0^5)], \end{aligned}$$

$$F_{[2]}^\theta = 0, \quad (5.62)$$

$$F_{[2]}^\phi = \frac{\dot{r}_0}{2\pi L^3 r_0^6 (L^2 + r_0^2)^{5/2}}(F_{\mathcal{E}[2]}^\phi \mathcal{E} + F_{\mathcal{K}[2]}^\phi \mathcal{K}), \quad (5.63)$$

where

$$\begin{aligned} F_{\mathcal{E}[2]}^\phi &= -(L^2 + r_0^2)(188L^8M + 406L^6Mr_0^2 - 64L^6r_0^3 + 222L^4Mr_0^4 - 163L^4r_0^5 - 145L^2r_0^7 - 48r_0^9) \\ &\quad - E^2L^2r_0^5(38L^4 + 31L^2r_0^2 - 15r_0^4), \\ F_{\mathcal{K}[2]}^\phi &= E^2L^2r_0^5(12L^4 + L^2r_0^2 - 15r_0^4) + r_0^2(L^2 + r_0^2)(94L^6M + 96L^4Mr_0^2 - 74L^4r_0^3 - 121L^2r_0^5 - 48r_0^7), \end{aligned}$$

$$F_{[4]}^t = \frac{3E\dot{r}_0}{40\pi r_0^{10}(r_0 - 2M)(L^2 + r^2)^{11/2}}(F_{\mathcal{E}[4]}^t \mathcal{E} + F_{\mathcal{K}[4]}^t \mathcal{K}), \quad (5.64)$$

where

$$\begin{aligned} F_{\mathcal{E}[4]}^t &= 30E^4r_0^{10}(64L^{10} + 384L^8r_0^2 + 989L^6r_0^4 + 1222L^4r_0^6 + 437L^2r_0^8 + 12r_0^{10}) \\ &\quad + 2E^2r_0^5(L^2 + r_0^2)(92160L^{12}M + 445008L^{10}Mr_0^2 + 859044L^8Mr_0^4 - 1920L^8r_0^5 + 838312L^6Mr_0^6 - 9780L^6r_0^7 \\ &\quad + 433114L^4Mr_0^8 - 19065L^4r_0^9 + 102188L^2Mr_0^{10} - 9870L^2r_0^{11} + 5030Mr_0^{12} - 585r_0^{13}) \\ &\quad + 4(L^2 + r_0^2)^2(46080L^{14}M^2 + 403200L^{12}M^2r_0^2 - 48000L^{12}Mr_0^3 + 1231984L^{10}M^2r_0^4 - 219840L^{10}Mr_0^5 \\ &\quad + 1841004L^8M^2r_0^6 - 411324L^8Mr_0^7 + 1490772L^6M^2r_0^8 - 406397L^6Mr_0^9 + 480L^6r_0^{10} + 668810L^4M^2r_0^{10} \\ &\quad - 232331L^4Mr_0^{11} + 2040L^4r_0^{12} + 161840L^2M^2r_0^{12} - 75367L^2Mr_0^{13} + 1590L^2r_0^{14} + 18382M^2r_0^{14} \\ &\quad - 10669Mr_0^{15} + 210r_0^{16}), \\ F_{\mathcal{K}[4]}^t &= -15E^4r_0^{12}(64L^8 + 328L^6r_0^2 + 495L^4r_0^4 + 22L^2r_0^6 - 81r_0^8) \\ &\quad - 4E^2r_0^7(L^2 + r_0^2)(25920L^{10}M + 102372L^8Mr_0^2 + 152523L^6Mr_0^4 - 480L^6r_0^5 + 103375L^4Mr_0^6 - 1575L^4r_0^7 \\ &\quad + 25889L^2Mr_0^8 - 120L^2r_0^9 - 455Mr_0^{10} + 495r_0^{11}) \\ &\quad - r_0^2(L^2 + r_0^2)^2(92160L^{12}M^2 + 766080L^{10}M^2r_0^2 - 130560L^{10}Mr_0^3 + 2014208L^8M^2r_0^4 - 497760L^8Mr_0^5 \\ &\quad + 2443016L^6M^2r_0^6 - 743088L^6Mr_0^7 + 1527236L^4M^2r_0^8 - 552712L^4Mr_0^9 + 960L^4r_0^{10} + 496596L^2M^2r_0^{10} \\ &\quad - 206180L^2Mr_0^{11} - 135L^2r_0^{12} + 73528M^2r_0^{12} - 30796Mr_0^{13} - 735r_0^{14}), \end{aligned}$$

$$F_{[4]}^r = \frac{3}{40\pi r_0^{14}(L^2 + r_0^2)^{11/2}}(F_{\mathcal{E}[4]}^r \mathcal{E} + F_{\mathcal{K}[4]}^r \mathcal{K}), \quad (5.65)$$

where

$$\begin{aligned}
F_{\mathcal{E}[4]}^r &= 30E^6 r_0^{13} (64L^{10} + 384L^8 r_0^2 + 989L^6 r_0^4 + 1222L^4 r_0^6 + 437L^2 r_0^8 + 12r_0^{10}) \\
&\quad + E^4 r_0^8 (L^2 + r_0^2) (184320L^{12} M + 893856L^{10} M r_0^2 - 1920L^{10} r_0^3 + 1746888L^8 M r_0^4 - 18240L^8 r_0^5 \\
&\quad + 1744604L^6 M r_0^6 - 53550L^6 r_0^7 + 907298L^4 M r_0^8 - 58665L^4 r_0^9 + 197536L^2 M r_0^{10} - 16320L^2 r_0^{11} + 9010M r_0^{12} \\
&\quad - 645r_0^{13}) \\
&\quad + 2E^2 r_0^3 (L^2 + r_0^2)^2 (92160L^{14} M^2 + 1036800L^{12} M^2 r_0^2 - 211200L^{12} M r_0^3 + 3363456L^{10} M^2 r_0^4 - 889424L^{10} M r_0^5 \\
&\quad + 5003472L^8 M^2 r_0^6 - 1487220L^8 M r_0^7 + 1920L^8 r_0^8 + 3866356L^6 M^2 r_0^8 - 1272840L^6 M r_0^9 + 9780L^6 r_0^{10} \\
&\quad + 1570564L^4 M^2 r_0^{10} - 594724L^4 M r_0^{11} + 10875L^4 r_0^{12} + 321308L^2 M^2 r_0^{12} - 146848L^2 M r_0^{13} + 1830L^2 r_0^{14} \\
&\quad + 36764M^2 r_0^{14} - 20288M r_0^{15} - 105r_0^{16}) \\
&\quad - (r_0 - 2M)(L^2 + r_0^2)^3 (184320L^{14} M^2 + 1711104L^{12} M^2 r_0^2 - 245760L^{12} M r_0^3 + 4872896L^{10} M^2 r_0^4 \\
&\quad - 884480L^{10} M r_0^5 + 6311728L^8 M^2 r_0^6 - 1185120L^8 M r_0^7 + 4083688L^6 M^2 r_0^8 - 721620L^6 M r_0^9 + 1920L^6 r_0^{10} \\
&\quad + 1299396L^4 M^2 r_0^{10} - 198700L^4 M r_0^{11} + 2460L^4 r_0^{12} + 209484L^2 M^2 r_0^{12} - 28120L^2 M r_0^{13} - 105L^2 r_0^{14} \\
&\quad + 28640M^2 r_0^{14} - 5120M r_0^{15} - 525r_0^{16}), \\
F_{\mathcal{K}[4]}^r &= -15E^6 r_0^{15} (64L^8 + 328L^6 r_0^2 + 495L^4 r_0^4 + 22L^2 r_0^6 - 81r_0^8) \\
&\quad - E^4 r_0^{10} (L^2 + r_0^2) (103680L^{10} M + 415248L^8 M r_0^2 - 2880L^8 r_0^3 + 627612L^6 M r_0^4 - 10680L^6 r_0^5 + 418570L^4 M r_0^6 \\
&\quad - 8835L^4 r_0^7 + 93896L^2 M r_0^8 + 4350L^2 r_0^9 - 2870M r_0^{10} + 2505r_0^{11}) \\
&\quad - E^2 r_0^5 (L^2 + r_0^2)^2 (92160L^{12} M^2 + 1019520L^{10} M^2 r_0^2 - 257280L^{10} M r_0^3 + 2806176L^8 M^2 r_0^4 - 893744L^8 M r_0^5 \\
&\quad + 3309168L^6 M^2 r_0^6 - 1180004L^6 M r_0^7 + 1920L^6 r_0^8 + 1878796L^4 M^2 r_0^8 - 724772L^4 M r_0^9 - 900L^4 r_0^{10} \\
&\quad + 517652L^2 M^2 r_0^{10} - 205608L^2 M r_0^{11} - 5685L^2 r_0^{12} + 73528M^2 r_0^{12} - 28696M r_0^{13} - 1785r_0^{14}) \\
&\quad + r_0^2 (r_0 - 2M)(L^2 + r_0^2)^3 (92160L^{12} M^2 + 815232L^{10} M^2 r_0^2 - 157440L^{10} M r_0^3 + 1939840L^8 M^2 r_0^4 \\
&\quad - 468160L^8 M r_0^5 + 1942488L^6 M^2 r_0^6 - 494560L^6 M r_0^7 + 892484L^4 M^2 r_0^8 - 217780L^4 M r_0^9 - 960L^4 r_0^{10} \\
&\quad + 195164L^2 M^2 r_0^{10} - 38820L^2 M r_0^{11} - 1545L^2 r_0^{12} + 28640M^2 r_0^{12} - 5120M r_0^{13} - 525r_0^{14}),
\end{aligned}$$

$$F_{[4]}^\theta = 0, \quad (5.66)$$

$$F_{[4]}^\phi = \frac{3\dot{r}_0}{40\pi r_0^{13} L^5 (L^2 + r^2)^{9/2}} (F_{\mathcal{E}[4]}^\phi \mathcal{E} + F_{\mathcal{K}[4]}^\phi \mathcal{K}), \quad (5.67)$$

where

$$\begin{aligned}
F_{\mathcal{E}[4]}^\phi &= 15E^4 L^4 r_0^{10} (128L^{10} + 960L^8 r_0^2 + 2266L^6 r_0^4 + 1369L^4 r_0^6 - 228L^2 r_0^8 - 35r_0^{10}) \\
&\quad + 2E^2 L^2 r_0^5 (L^2 + r_0^2) (115200L^{14} M + 449744L^{12} M r_0^2 + 660732L^{10} M r_0^4 - 5760L^{10} r_0^5 + 442406L^8 M r_0^6 \\
&\quad - 21300L^8 r_0^7 + 116472L^6 M r_0^8 - 18475L^6 r_0^9 - 1186L^4 M r_0^{10} + 4310L^4 r_0^{11} + 10765L^2 r_0^{13} + 4240r_0^{15}) \\
&\quad + (L^2 + r_0^2)^2 (184320L^{18} M^2 + 1711104L^{16} M^2 r_0^2 - 245760L^{16} M r_0^3 + 4872896L^{14} M^2 r_0^4 - 884480L^{14} M r_0^5 \\
&\quad + 6311728L^{12} M^2 r_0^6 - 1090720L^{12} M r_0^7 + 4083688L^{10} M^2 r_0^8 - 436180L^{10} M r_0^9 - 5760L^{10} r_0^{10} \\
&\quad + 1299396L^8 M^2 r_0^{10} + 73140L^8 M r_0^{11} - 83700L^8 r_0^{12} + 209484L^6 M^2 r_0^{12} + 15720L^6 M r_0^{13} - 236585L^6 r_0^{14} \\
&\quad + 28640L^4 M^2 r_0^{14} - 63200L^4 M r_0^{15} - 271325L^4 r_0^{16} - 21120L^2 M r_0^{17} - 138400L^2 r_0^{18} - 25600r_0^{20}), \\
F_{\mathcal{K}[4]}^\phi &= -15E^4 L^4 r_0^{12} (192L^8 + 584L^6 r_0^2 + 169L^4 r_0^4 - 322L^2 r_0^6 - 35r_0^8) \\
&\quad - 2E^2 L^2 r_0^7 (L^2 + r_0^2) (63360L^{12} M + 197992L^{10} M r_0^2 + 216538L^8 M r_0^4 - 4800L^8 r_0^5 + 87890L^6 M r_0^6 - 7110L^6 r_0^7 \\
&\quad + 5264L^4 M r_0^8 + 2455L^4 r_0^9 + 8645L^2 r_0^{11} + 4240r_0^{13}) \\
&\quad - r_0^2 (L^2 + r_0^2)^2 (92160L^{16} M^2 + 815232L^{14} M^2 r_0^2 - 157440L^{14} M r_0^3 + 1939840L^{12} M^2 r_0^4 - 422080L^{12} M r_0^5 \\
&\quad + 1942488L^{10} M^2 r_0^6 - 309120L^{10} M r_0^7 + 892484L^8 M^2 r_0^8 + 9580L^8 M r_0^9 - 39360L^8 r_0^{10} + 195164L^6 M^2 r_0^{10} \\
&\quad + 22780L^6 M r_0^{11} - 152745L^6 r_0^{12} + 28640L^4 M^2 r_0^{12} - 52640L^4 M r_0^{13} - 213325L^4 r_0^{14} - 21120L^2 M r_0^{15} \\
&\quad - 125600L^2 r_0^{16} - 25600r_0^{18}).
\end{aligned}$$

2. *huu regularization*

The quantity

$$H^{(R)} = \frac{1}{2} h_{ab}^{(R)} u^a u^b \quad (5.68)$$

was first proposed by Detweiler [5] as a tool for constructing gauge invariant measurements from self-force calculations. It has since proven invaluable in extracting gauge invariant results from gauge dependent self-force calculations [61, 77].

Much the same as with self-force calculations, the calculation of $H^{(R)}$ requires the subtraction of the appropriate singular piece, $H^{(S)} = \frac{1}{2} h_{ab}^{(S)} u^a u^b$ from the full retarded field. In this section, we give this subtraction in the form of mode-sum regularization parameters. In doing so, we keep with our convention that the term proportional to $l + \frac{1}{2}$ is denoted by $H_{[-1]}$ ($= 0$ in this case), the constant term is denoted by $H_{[0]}$, and so on.

Note that, as in the self-force case, an ambiguity arises here due to the presence of terms involving the four-velocity at x . One is free to arbitrarily choose how to define this provided $\lim_{x \rightarrow \bar{x}} u^a = u^{\bar{a}}$. As before, we choose this in such a way that the Schwarzschild components of the four velocity at x are exactly those at \bar{x} . The regularisation parameters are then given by

$$H_{[0]} = \frac{2\mathcal{K}}{\pi\sqrt{L^2 + r_0^2}}, \quad (5.69)$$

$$H_{[2]} = \frac{H_{[2]}^{\mathcal{E}} \mathcal{E} + H_{[2]}^{\mathcal{K}} \mathcal{K}}{\pi r_0^3 (L^2 + r_0^2)^{3/2}}, \quad (5.70)$$

where

$$\begin{aligned} H_{[2]}^{\mathcal{E}} &= 2E^2 r_0^5 + (L^2 + r_0^2)(36L^2 M - 8L^2 r_0 + 38Mr_0^2 - 9r_0^3), \\ H_{[2]}^{\mathcal{K}} &= -E^2 r_0^3 (16L^2 + 17r_0^2) - 2(L^2 + r_0^2)(16L^2 M - 4L^2 r_0 + 33Mr_0^2 - 12r_0^3), \end{aligned}$$

$$H_{[4]} = \frac{3(H_{[4]}^{\mathcal{E}} \mathcal{E} + H_{[4]}^{\mathcal{K}} \mathcal{K})}{20\pi r^{10} (L^2 + r^2)^{7/2}}, \quad (5.71)$$

where

$$\begin{aligned} H_{[4]}^{\mathcal{E}} &= -120E^4 r_0^{12} (8L^4 + 17L^2 r_0^2 + 7r_0^4) + 2E^2 r_0^5 (L^2 + r_0^2) (3584L^8 M + 12712L^6 Mr_0^2 + 15516L^4 Mr_0^4 \\ &\quad + 120L^4 r_0^5 + 6182L^2 Mr_0^6 + 735L^2 r_0^7 + 34Mr_0^8 + 495r_0^9) \\ &\quad + 2(L^2 + r_0^2)^2 (1536L^{10} M^2 + 13888L^8 M^2 r_0^2 - 1600L^8 Mr_0^3 + 40584L^6 M^2 r_0^4 - 9440L^6 Mr_0^5 + 46888L^4 M^2 r_0^6 \\ &\quad - 14100L^4 Mr_0^7 + 120L^4 r_0^8 + 18936L^2 M^2 r_0^8 - 5350L^2 Mr_0^9 + 15L^2 r_0^{10} + 340M^2 r_0^{10} + 850Mr_0^{11} - 90r_0^{12}), \\ H_{[4]}^{\mathcal{K}} &= 15E^4 r_0^{10} (64L^6 + 224L^4 r_0^2 + 259L^2 r_0^4 + 91r_0^6) \\ &\quad - 4E^2 r_0^7 (L^2 + r_0^2) (1376L^6 M + 3174L^4 Mr_0^2 + 420L^4 r_0^3 + 1965L^2 Mr_0^4 + 960L^2 r_0^5 + 227Mr_0^6 + 510r_0^7) \\ &\quad - r_0^2 (L^2 + r_0^2)^2 (1536L^8 M^2 + 15904L^6 M^2 r_0^2 - 7360L^6 Mr_0^3 + 36160L^4 M^2 r_0^4 - 19320L^4 Mr_0^5 + 22412L^2 M^2 r_0^6 \\ &\quad - 11040L^2 Mr_0^7 - 720L^2 r_0^8 + 680M^2 r_0^8 + 860Mr_0^9 - 705r_0^{10}). \end{aligned}$$

F. Example

As an example application of our high order regularization parameters, we consider the case of a scalar particle on a circular geodesic of the Schwarzschild spacetime. In this case, the retarded field may be computed using the frequency domain method described in [53], along with improved asymptotics for the boundary conditions (by expanding inside the exponential rather than outside) and with the use of the arbitrary precision differential equation solving support in *Mathematica* [78]. These improvements allowed us to substantially increase the accuracy of the computed retarded field. We found this to be necessary to get the full benefit from the the higher order regularization parameters.

In Fig. 2, we show the effect of subtracting in turn the cumulative sums of the regularization parameters from the full retarded field. Shown in the figure in order from top to bottom are F_r^{ret} and the result of subtracting from it in

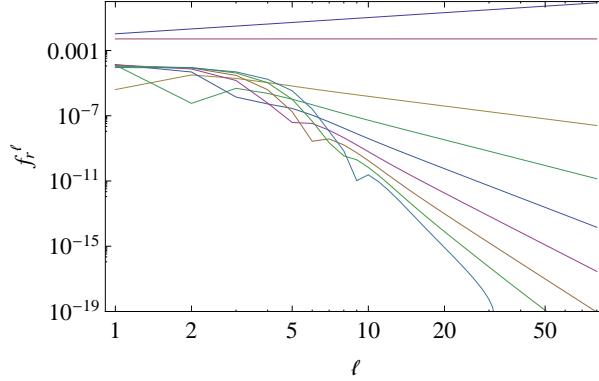


FIG. 2. Regularization of the radial component of the self-force for the case of a scalar particle on a circular geodesic of radius $r_0 = 10M$ in Schwarzschild spacetime. In decreasing slope the above lines represent the unregularised self-force, self-force regularised by subtracting from it in turn the cumulative sum of $A_r^l, B_r^l, D_r^l, F_r^l, H_r^l, J_r^l, L_r^l, N_r^l$.

turn the cumulative sum of the regularization terms $A_r^l, B_r^l, D_r^l, F_r^l, H_r^l, J_r^l, L_r^l$ and N_r^l . The parameters A_r, B_r, D_r, F_r and H_r are the analytically derived ones given in Sec. V C. The parameters J_r, L_r and H_r were determined through a numerical fit to the data. The resulting rapid convergence with l enables the calculation of an extremely accurate value for the self-force. Summing over l , we find $F_r = 0.000013784482575667959(3)$, where the uncertainty in the last digit is estimated by assuming that the error comes purely from the fact that the sum is only done up to a finite $l_{\max} = 80$.

VI. EFFECTIVE SOURCE

As another application of our high order expansions of the Detweiler-Whiting singular field, we consider its use in the effective source approach to calculating the self-force. The effective source approach – independently proposed by Barack and Golbourn [37] and by Vega and Detweiler [38] – relies on knowledge of the singular field to derive an equation for a regularized field that gives the self-force without any need for post-processed regularization. If the singular field is known exactly, then the regularized field is totally regular and is a solution of the homogeneous wave equation. In reality, exact expressions for the singular field can only be obtained for very simple spacetimes. More generally, the best one can do is an approximation such as that given in Sec. III. Splitting the retarded field into approximate singular and regularized parts,

$$\varphi_{(\text{ret})}^A = \tilde{\varphi}_{(\text{S})}^A + \tilde{\varphi}_{(\text{R})}^A, \quad (6.1)$$

substituting into the wave equation, Eq. (2.1), and rearranging, we obtain an equation for the regularized field,

$$\mathcal{D}^A{}_B \tilde{\varphi}_{(\text{R})}^B = S_{\text{eff}}^A, \quad (6.2)$$

with an *effective source*,

$$S_{\text{eff}}^A = -\mathcal{D}^A{}_B \tilde{\varphi}_{(\text{S})}^B - 4\pi \mathcal{Q} \int u^A \delta_4(x, z(\tau')) d\tau'. \quad (6.3)$$

For sufficiently good approximations to the singular field, $\tilde{\varphi}_{(\text{R})}^A$ and S^A are finite everywhere, in particular on the world-line. As a result, one never encounters problematic singularities or δ -functions, making the approach particularly suitable for use in time domain numerical simulations. A detailed review of this approach can be found in [79, 80].

In Figs. 3, 4, 5 and 6 we show the result of applying our expansions to the case of a scalar particle on a circular geodesic of radius $r_0 = 10M$ in Schwarzschild spacetime. Similar plots can be obtained for the electromagnetic case, gravitational case and for more generic motion. However, the general structure does not change and is best illustrated by this simple example.

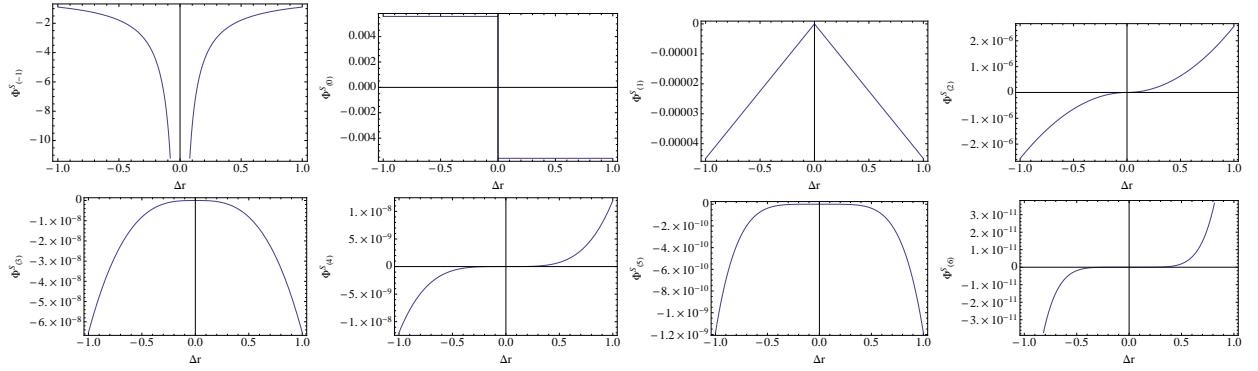


FIG. 3. Terms in the coordinate expansion of the singular field for $\mathcal{O}(\epsilon^{-1})$ (top left) to $\mathcal{O}(\epsilon^6)$ (bottom right). Shown is the case of a circular geodesic of radius $r_0 = 10M$ in Schwarzschild spacetime.

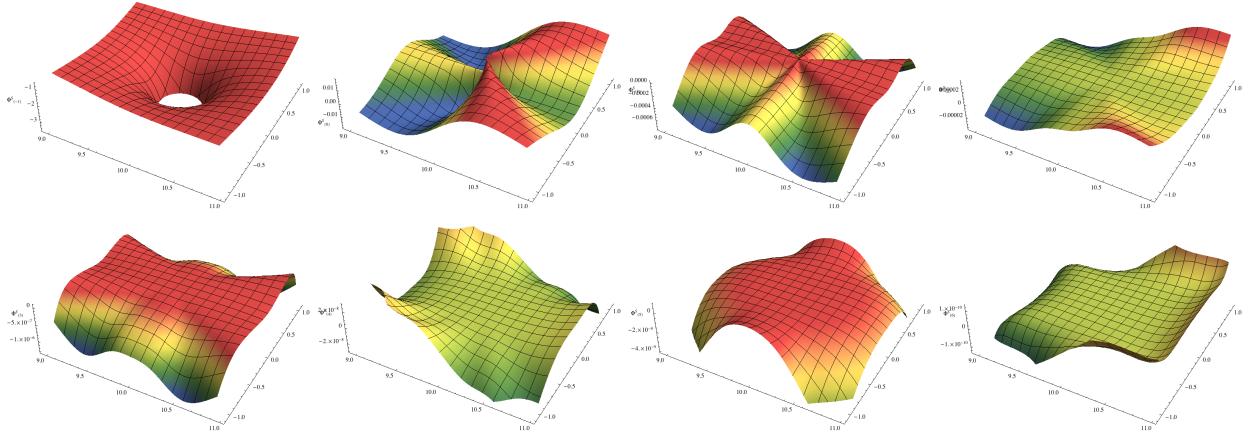


FIG. 4. Terms in the coordinate expansion of the singular field for $\mathcal{O}(\epsilon^{-1})$ (top left) to $\mathcal{O}(\epsilon^6)$ (bottom right). Shown is the case of a circular geodesic of radius $r_0 = 10M$ in Schwarzschild spacetime.

VII. DISCUSSION

In this paper we have developed high order expansions of the Detweiler-Whiting singular field of a point scalar or electromagnetic particle and of a point mass. Many of our expressions are very general, not necessarily being restricted to any particular choice of spacetime. In our explicit coordinate calculations, however, for simplicity we

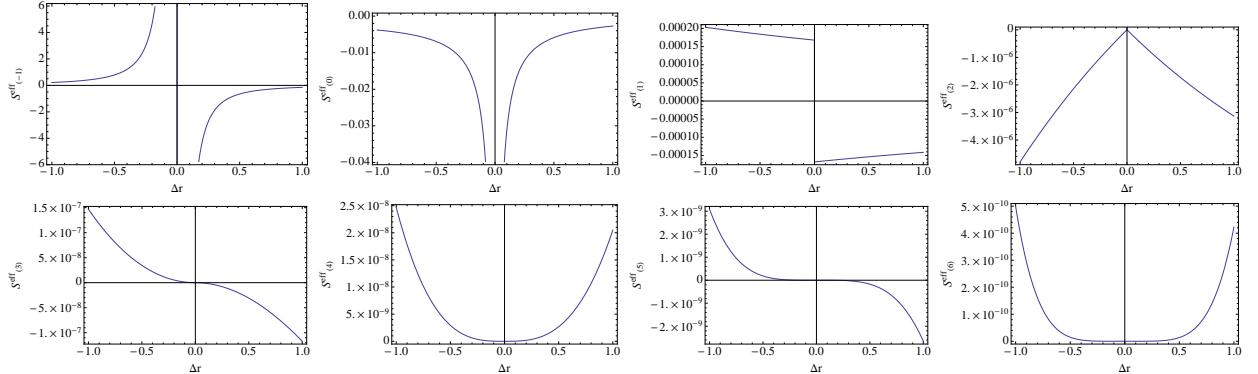


FIG. 5. Effective source for the approximate singular field of order $\mathcal{O}(\epsilon^{-1})$ (top left) to $\mathcal{O}(\epsilon^6)$ (bottom right). Shown is the case of a circular geodesic of radius $r_0 = 10M$ in Schwarzschild spacetime.

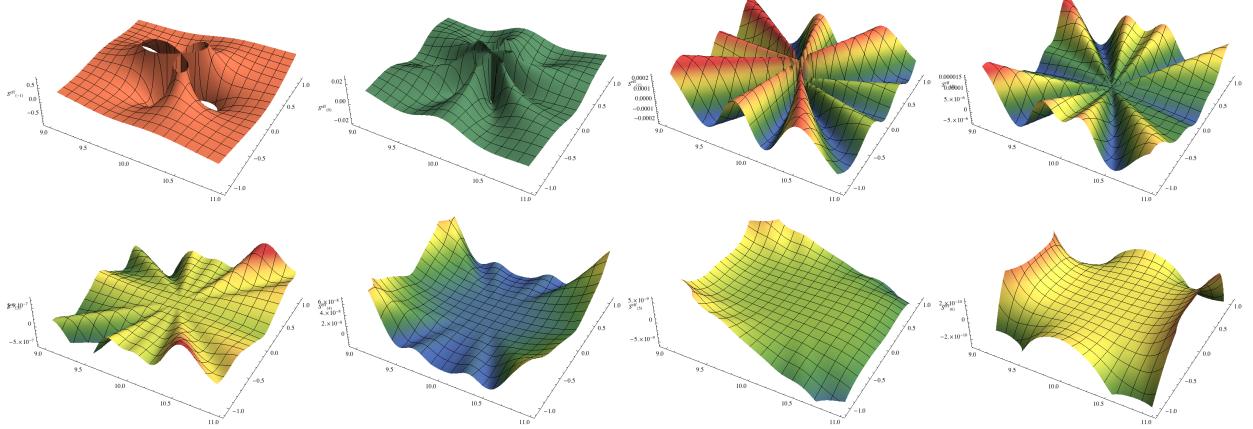


FIG. 6. Effective source for the approximate singular field of order $\mathcal{O}(\epsilon^{-1})$ (top left) to $\mathcal{O}(\epsilon^6)$ (bottom right). Shown is the case of a circular geodesic of radius $r_0 = 10M$ in Schwarzschild spacetime.

chose to limit ourselves to the Schwarzschild geometry. A logical extension of this work would be to consider, instead, the Kerr spacetime. This has already been done to $\mathcal{O}(\epsilon^2)$ for the case of a scalar charge in a circular geodesic orbit in Kerr spacetime and spherical-harmonic regularization parameters have been computed [81]. We will present a more general version of this calculation in forthcoming work [82]. However, the use of spherical harmonics is not well suited to the Kerr spacetime. A more appropriate choice of basis functions are the spheroidal harmonics; it may be more sensible to compute regularization parameters in this spheroidal harmonic basis. Nevertheless, the full singular field is the same, independently of the choice of basis functions, and the majority of our calculations can easily be carried over to the Kerr case with little modification.

In the cases of electromagnetic and gravitational fields, there is an arbitrariness in the choice of gauge. For this work, we have focused only on the case of Lorenz gauge, $A^a_{;a} = 0$ and $h_{ab}^{;b} = 0$, where the singular field is best understood. However, other gauges such as Regge-Wheeler and Radiation gauge are better suited to time domain numerical calculations. Indeed, there are many outstanding problems in the time domain evolution of metric perturbations in Lorenz gauge [83]. Given recent developments in understanding the singular behavior of the retarded field in other gauges [72, 73, 84, 85], it may soon be possible to implement calculations similar to what we have presented here for other gauges. This could be achieved, for example, by either taking our Lorenz gauge expressions and performing the gauge transformation to a different gauge or by directly computing the singular field in the other gauge. It should be noted, however, that there are many non-trivial issues which can arise when considering other gauges; for example, the gauge transformation itself may be singular or the singular field may be even more singular² than in Lorenz gauge. Any calculations will require considerable care to correctly deal with such issues.

In Sec. VI we gave plots of an effective source which may be used in a numerical evolution. We have not yet, however, fed this effective source into an actual numerical code. Existing results from effective source calculations have used a source obtained by taking the first four terms in the expansion of the singular field (to $\mathcal{O}(\epsilon^2)$) which yields a source which is C^0 . This is sufficient to obtain correct results, but limits the convergence of any numerical scheme with resolution or number of modes required. By including the next four terms in the expansion of the singular field (to $\mathcal{O}(\epsilon^6)$) the highest order effective source shown in Sec. VI is now C^4 , allowing for the use of high-order finite differencing to potentially obtain a significant increase in numerical accuracy.

We have limited the focus of the present work to linear order. However, for calculations with sufficient accuracy to match the requirements of gravitational wave detection, it will be necessary to go beyond first order and to compute a second order self-force. Rapid progress is being made in resolving the many conceptual issues which arise when trying to derive the second order equations of motion [29–31] and the computation of a second order effective source. The next step is to implement this second order prescription in a numerical calculation. As this involves an effective source which contains terms involving the first order singular field, it seems likely that many of the calculations done here may be of use for second order.

² A specific example of this is the case of metric perturbations in Regge-Wheeler gauge, where there are additional delta-function divergences in the singular field which are not smoothed out by a mode decomposition [71].

ACKNOWLEDGEMENTS

We are extremely grateful to Niels Warburton, Sarp Akcay and Leor Barack for making available their data for the spherical harmonic modes of the retarded field in scalar and gravitational cases, and to Roland Hass for making the equivalent data available for the electromagnetic case. We thank Sam Dolan, Marc Casals, José Luis Jaramillo, Michael Jasiulek, Abraham Harte and particularly Ian Vega for many insightful discussions during the progress of this work. Finally, we thank participants of the 2010 and 2011 Capra meetings (in Waterloo and Southampton, respectively) for many illuminating conversations.

Appendix A: Coordinate Expansions in Schwarzschild Space-Time

In this Appendix we give coordinate expansions of the key quantities appearing in the singular field, Eqs. (2.13), (2.18) and (2.23).

Synge World function

Letting $\cos \gamma = \cos \theta \cos \theta' - \sin \theta \sin \theta' \cos(\phi - \phi')$ so that $2(1 - \cos \gamma) = \Delta w_1^2 + \Delta w_2^2$, the expansion of the world function to the order required in this paper is

$$\sigma(x, x') = \sum_{i,j,k=0}^{i+j+2k \leq 9} \sigma_{ijk} (t' - t)^i (r' - r)^j (1 - \cos \gamma)^k + \mathcal{O}(\epsilon^{10}), \quad (\text{A1})$$

where the non-zero coefficients are

$$\begin{aligned} \sigma_{001} &= r^2, \quad \sigma_{002} = \frac{Mr}{3}, \quad \sigma_{003} = \frac{1}{90} M(9r - 2M), \quad \sigma_{004} = \frac{M(14M^2 - 19Mr + 30r^2)}{840r}, \quad \sigma_{020} = \frac{M}{r} - \frac{1}{2}, \\ \sigma_{021} &= \frac{M(r - 2M)}{6r^2}, \quad \sigma_{022} = \frac{M(10M^2 - 11Mr + 3r^2)}{60r^3}, \quad \sigma_{023} = \frac{M(-92M^3 + 142M^2r - 78Mr^2 + 15r^3)}{840r^4}, \\ \sigma_{040} &= \frac{M^2(2M - r)}{24r^5}, \quad \sigma_{041} = -\frac{M^2(54M^2 - 49Mr + 11r^2)}{360r^6}, \\ \sigma_{042} &= \frac{M^2(1956M^3 - 2702M^2r + 1228Mr^2 - 183r^3)}{10080r^7}, \quad \sigma_{060} = \frac{M^3(26M^2 - 25Mr + 6r^2)}{720r^9}, \\ \sigma_{061} &= \frac{M^3(-1818M^3 + 2475M^2r - 1115Mr^2 + 166r^3)}{15120r^{10}}, \quad \sigma_{080} = \frac{M^4(978M^3 - 1393M^2r + 660Mr^2 - 104r^3)}{40320r^{13}}, \\ \sigma_{101} &= r, \quad \sigma_{102} = \frac{M}{6}, \quad \sigma_{103} = \frac{M}{20}, \quad \sigma_{104} = \frac{1}{840} M \left(15 - \frac{7M^2}{r^2} \right), \quad \sigma_{120} = -\frac{M}{2r^2}, \quad \sigma_{121} = \frac{M(4M - r)}{12r^3}, \\ \sigma_{122} &= -\frac{M(30M^2 - 22Mr + 3r^2)}{120r^4}, \quad \sigma_{123} = \frac{M(368M^3 - 426M^2r + 156Mr^2 - 15r^3)}{1680r^5}, \quad \sigma_{140} = \frac{M^2(2r - 5M)}{24r^6}, \\ \sigma_{141} &= \frac{M^2(324M^2 - 245Mr + 44r^2)}{720r^7}, \quad \sigma_{142} = \frac{M^2(-3423M^3 + 4053M^2r - 1535Mr^2 + 183r^3)}{5040r^8}, \\ \sigma_{160} &= -\frac{M^3(117M^2 - 100Mr + 21r^2)}{720r^{10}}, \quad \sigma_{161} = \frac{M^3(18180M^3 - 22275M^2r + 8920Mr^2 - 1162r^3)}{30240r^{11}}, \\ \sigma_{180} &= \frac{M^4(-6357M^3 + 8358M^2r - 3630Mr^2 + 520r^3)}{40320r^{14}}, \quad \sigma_{200} = -\frac{r}{4M - 2r}, \quad \sigma_{201} = \frac{M}{12M - 6r}, \\ \sigma_{202} &= \frac{M(r - M)}{60r(2M - r)}, \quad \sigma_{203} = \frac{M(-14M^2 + 6Mr + 3r^2)}{840r^2(2M - r)}, \quad \sigma_{220} = \frac{M(5M - 4r)}{12r^3(2M - r)}, \\ \sigma_{221} &= -\frac{M(23M^2 - 20Mr + 3r^2)}{60r^4(2M - r)}, \quad \sigma_{222} = \frac{M(686M^3 - 802M^2r + 281Mr^2 - 24r^3)}{1680r^5(2M - r)}, \\ \sigma_{240} &= \frac{M^2(429M^2 - 394Mr + 86r^2)}{720r^7(2M - r)}, \quad \sigma_{241} = \frac{M^2(-7509M^3 + 9170M^2r - 3575Mr^2 + 438r^3)}{5040r^8(2M - r)}, \end{aligned}$$

$$\begin{aligned}
\sigma_{260} &= \frac{M^3 (23931M^3 - 31560M^2r + 13652Mr^2 - 1930r^3)}{30240r^{11}(2M - r)}, \quad \sigma_{300} = -\frac{M}{2(r - 2M)^2}, \quad \sigma_{301} = \frac{M}{12(r - 2M)^2}, \\
\sigma_{302} &= \frac{M (2M^2 - 2Mr + r^2)}{120r^2(r - 2M)^2}, \quad \sigma_{303} = \frac{M (56M^3 - 54M^2r + 12Mr^2 + 3r^3)}{1680r^3(r - 2M)^2}, \quad \sigma_{320} = -\frac{M(M - r)^2}{4r^4(r - 2M)^2}, \\
\sigma_{321} &= \frac{M (24M^3 - 55M^2r + 32Mr^2 - 4r^3)}{120r^5(r - 2M)^2}, \quad \sigma_{322} = \frac{M (-630M^4 + 1234M^3r - 832M^2r^2 + 218Mr^3 - 15r^4)}{1680r^6(r - 2M)^2}, \\
\sigma_{340} &= \frac{M^2 (-301M^3 + 460M^2r - 228Mr^2 + 36r^3)}{240r^8(r - 2M)^2}, \\
\sigma_{341} &= \frac{M^2 (35472M^4 - 63269M^3r + 40912M^2r^2 - 11240Mr^3 + 1088r^4)}{10080r^9(r - 2M)^2}, \\
\sigma_{360} &= \frac{M^3 (-27687M^4 + 51002M^3r - 34745M^2r^2 + 10344Mr^3 - 1131r^4)}{10080r^{12}(r - 2M)^2}, \quad \sigma_{400} = -\frac{M(M - 8r)}{24r(r - 2M)^3}, \\
\sigma_{401} &= \frac{M (5M^2 - 3Mr - 6r^2)}{120r^2(r - 2M)^3}, \quad \sigma_{402} = \frac{M (42M^3 - 70M^2r + 39Mr^2 - 16r^3)}{3360r^3(r - 2M)^3}, \\
\sigma_{420} &= -\frac{M (M^3 - 78M^2r + 116Mr^2 - 48r^3)}{240r^5(r - 2M)^3}, \quad \sigma_{421} = -\frac{M (553M^4 - 269M^3r - 444M^2r^2 + 322Mr^3 - 40r^4)}{1680r^6(r - 2M)^3}, \\
\sigma_{440} &= -\frac{M^2 (42507M^4 - 94876M^3r + 77976M^2r^2 - 27740Mr^3 + 3546r^4)}{20160r^9(r - 2M)^3}, \quad \sigma_{500} = -\frac{M (M^2 - 2Mr + 6r^2)}{24r^2(r - 2M)^4}, \\
\sigma_{501} &= \frac{M (20M^3 - 31M^2r + 12Mr^2 + 8r^3)}{240r^3(r - 2M)^4}, \quad \sigma_{502} = \frac{M (7M^4 - 21M^3r + 23M^2r^2 - 11Mr^3 + 5r^4)}{1680r^4(r - 2M)^4}, \\
\sigma_{520} &= \frac{M (75M^4 - 84M^3r - 51M^2r^2 + 104Mr^3 - 40r^4)}{240r^6(r - 2M)^4}, \\
\sigma_{521} &= \frac{M (-4396M^5 + 8227M^4r - 4760M^3r^2 + 350M^2r^3 + 412Mr^4 - 60r^5)}{3360r^7(r - 2M)^4}, \\
\sigma_{540} &= \frac{M^2 (-57987M^5 + 178306M^4r - 214952M^3r^2 + 126744M^2r^3 - 36328Mr^4 + 3992r^5)}{20160r^{10}(r - 2M)^4}, \\
\sigma_{600} &= -\frac{M (35M^3 - 86M^2r + 86Mr^2 - 144r^3)}{720r^3(r - 2M)^5}, \quad \sigma_{601} = \frac{M (245M^4 - 532M^3r + 421M^2r^2 - 120Mr^3 - 40r^4)}{1680r^4(r - 2M)^5}, \\
\sigma_{620} &= \frac{M (2317M^5 - 5560M^4r + 4220M^3r^2 - 146M^2r^3 - 1254Mr^4 + 480r^5)}{3360r^7(r - 2M)^5}, \\
\sigma_{700} &= \frac{M (-15M^4 + 44M^3r - 54M^2r^2 + 36Mr^3 - 40r^4)}{240r^4(r - 2M)^6}, \\
\sigma_{701} &= \frac{M (280M^5 - 763M^4r + 832M^3r^2 - 444M^2r^3 + 100Mr^4 + 20r^5)}{1120r^5(r - 2M)^6}, \\
\sigma_{720} &= \frac{M (3759M^6 - 12402M^5r + 16015M^4r^2 - 9336M^3r^3 + 1347M^2r^4 + 1048Mr^5 - 420r^6)}{3360r^8(r - 2M)^6}, \\
\sigma_{800} &= \frac{M (385M^5 - 1316M^4r + 1928M^3r^2 - 1576M^2r^3 + 788Mr^4 - 640r^5)}{4480r^5(2M - r)^7}, \\
\sigma_{900} &= \frac{M (-5005M^6 + 19558M^5r - 33394M^4r^2 + 32584M^3r^3 - 19960M^2r^4 + 7984Mr^5 - 5040r^6)}{40320r^6(r - 2M)^8}. \tag{A2}
\end{aligned}$$

Van Vleck determinant

Inserting the above expansion for $\sigma(x, x')$ into the definition of the Van Vleck-Morette determinant, Eq. (3.2), gives

$$\Delta^{1/2}(x, x') = 1 + \sum_{i,j,k=0}^{i+j+2k \leq 7} \Delta_{ijk}^{1/2} (t' - t)^i (r' - r)^j (1 - \cos \gamma)^k + \mathcal{O}(\epsilon^{10}), \quad (\text{A3})$$

where the non-zero coefficients are

$$\begin{aligned} \Delta_{002}^{1/2} &= \frac{M^2}{15r^2}, \quad \Delta_{003}^{1/2} = \frac{M^2(27r - 34M)}{378r^3}, \quad \Delta_{021}^{1/2} = \frac{M^2(r - 2M)}{30r^5}, \quad \Delta_{022}^{1/2} = \frac{M^2(460M^2 - 428Mr + 99r^2)}{2520r^6}, \\ \Delta_{040}^{1/2} &= \frac{M^2(r - 2M)^2}{60r^8}, \quad \Delta_{041}^{1/2} = \frac{M^2(-1300M^3 + 1682M^2r - 714Mr^2 + 99r^3)}{5040r^9}, \\ \Delta_{060}^{1/2} &= \frac{M^3(199M - 89r)(r - 2M)^2}{7560r^{12}}, \quad \Delta_{102}^{1/2} = -\frac{M^2}{15r^3}, \quad \Delta_{103}^{1/2} = \frac{M^2(17M - 9r)}{126r^4}, \quad \Delta_{121}^{1/2} = \frac{M^2(5M - 2r)}{30r^6}, \\ \Delta_{122}^{1/2} &= -\frac{M^2(690M^2 - 535Mr + 99r^2)}{1260r^7}, \quad \Delta_{140}^{1/2} = -\frac{M^2(16M^2 - 14Mr + 3r^2)}{60r^9}, \\ \Delta_{141}^{1/2} &= \frac{M^2(5850M^3 - 6728M^2r + 2499Mr^2 - 297r^3)}{5040r^{10}}, \quad \Delta_{160}^{1/2} = \frac{M^3(-3184M^3 + 4224M^2r - 1850Mr^2 + 267r^3)}{5040r^{13}}, \\ \Delta_{201}^{1/2} &= \frac{M^2}{60Mr^3 - 30r^4}, \quad \Delta_{202}^{1/2} = \frac{M^2(322M - 177r)}{2520r^4(2M - r)}, \quad \Delta_{220}^{1/2} = -\frac{M^2}{30r^6}, \quad \Delta_{221}^{1/2} = -\frac{M^2(443M^2 - 412Mr + 90r^2)}{1260r^7(2M - r)}, \\ \Delta_{240}^{1/2} &= \frac{M^2(5648M^2 - 4800Mr + 981r^2)}{10080r^{10}}, \quad \Delta_{301}^{1/2} = \frac{M^2(2r - 3M)}{30r^4(r - 2M)^2}, \quad \Delta_{302}^{1/2} = \frac{M^2(-308M^2 + 332Mr - 93r^2)}{1260r^5(r - 2M)^2}, \\ \Delta_{320}^{1/2} &= \frac{M^2}{10r^7}, \quad \Delta_{321}^{1/2} = \frac{M^2(161M^3 - 464M^2r + 317Mr^2 - 60r^3)}{1260r^8(r - 2M)^2}, \quad \Delta_{340}^{1/2} = -\frac{M^2(2020M^2 - 1872Mr + 393r^2)}{2520r^{11}}, \\ \Delta_{400}^{1/2} &= \frac{M^2}{60r^4(r - 2M)^2}, \quad \Delta_{401}^{1/2} = -\frac{M^2(910M^2 - 1268Mr + 459r^2)}{5040r^5(r - 2M)^3}, \\ \Delta_{420}^{1/2} &= \frac{M^2(-3526M^2 + 3746Mr - 981r^2)}{5040r^8(r - 2M)^2}, \quad \Delta_{500}^{1/2} = \frac{M^2(4M - 3r)}{60r^5(r - 2M)^3}, \\ \Delta_{501}^{1/2} &= \frac{M^2(-1190M^3 + 2664M^2r - 2035Mr^2 + 537r^3)}{5040r^6(r - 2M)^4}, \\ \Delta_{520}^{1/2} &= -\frac{M^2(9392M^3 - 15628M^2r + 8631Mr^2 - 1572r^3)}{5040r^9(r - 2M)^3}, \quad \Delta_{600}^{1/2} = \frac{M^2(5432M^2 - 7720Mr + 2943r^2)}{30240r^6(r - 2M)^4}, \\ \Delta_{700}^{1/2} &= \frac{M^2(1036M^3 - 2120M^2r + 1524Mr^2 - 393r^3)}{2520r^7(r - 2M)^5}. \end{aligned} \quad (\text{A4})$$

Expansions of an arbitrary point on the world line about $x^{\bar{a}}$

The four velocity of a general geodesic orbit taken to lie in the equatorial plane is given by the standard expressions [86]

$$\dot{t}(\tau) = \frac{Er(\tau)}{r(\tau) - 2M}, \quad \dot{r}(\tau) = \sqrt{E^2 - \left(1 - \frac{2M}{r(\tau)}\right)\left(1 + \frac{L^2}{r(\tau)^2}\right)}, \quad \dot{\theta}(\tau) = 0, \quad \dot{\phi}(\tau) = \frac{L}{r(\tau)^2}. \quad (\text{A5})$$

It is straightforward to calculate the higher order proper time derivatives of these expressions and evaluate both the four velocity and its higher derivatives at $x^{\bar{a}}$, giving, for example,

$$\begin{aligned} \ddot{t}_0 &= \frac{Er_0}{r_0 - 2M}, \quad \dot{r}_0 = \sqrt{E^2 - \left(1 - \frac{2M}{r_0}\right)\left(1 + \frac{L^2}{r_0^2}\right)}, \quad \dot{\theta}_0 = 0, \quad \dot{\phi}_0 = \frac{L}{r_0^2}, \\ \ddot{r}_0 &= -\frac{2EM\dot{r}_0}{(r_0 - 2M)^2}, \quad \ddot{\theta}_0 = \frac{L^2r_0 - Mr_0^2 - 3L^2M}{r_0^4}, \quad \ddot{\phi}_0 = -\frac{2L\dot{r}_0}{r_0^3}, \end{aligned}$$

$$\begin{aligned}\ddot{t}_0 &= \frac{2EM [2(E^2 - 1)r_0^4 - r_0^2(3L^2 + 2M^2) + 9L^2Mr_0 + 5Mr_0^3 - 6L^2M^2]}{r_0^4(r_0 - 2M)^3}, \\ \ddot{r}_0 &= \frac{\dot{r}_0(-3L^2r_0 + 2Mr_0^2 + 12L^2M)}{r_0^5}, \\ \ddot{\theta}_0 &= 0, \quad \ddot{\phi}_0 = \frac{2L[3(E^2 - 1)r_0^3 - 4L^2r_0 + 7Mr_0^2 + 9L^2M]}{r_0^7}.\end{aligned}\tag{A6}$$

Combining Eq. (A6) with Eq. (3.17), we can express $x^{a'}$ in terms of $x^{\bar{a}}$ and $\Delta\tau$:

$$\begin{aligned}t' &= \frac{Er_0}{r_0 - 2M}\Delta\tau - \frac{EM\dot{r}_0}{(r_0 - 2M)^2}\Delta\tau^2 + \dots, \quad r' = r_0 + \dot{r}_0\Delta\tau - \frac{(-L^2r_0 + Mr_0^2 + 3L^2M)}{2r_0^4}\Delta\tau^2 + \dots, \\ \theta' &= \frac{\pi}{2}, \quad \phi' = \frac{L}{r_0^2}\Delta\tau - \frac{L\dot{r}_0}{r_0^3}\Delta\tau^2 + \dots.\end{aligned}\tag{A7}$$

It is also straightforward to obtain $\delta x^{a'}$, in terms of Δx^a , $x^{\bar{a}}$ and $\Delta\tau$ by noting that $\delta x^{a'} = x^{a'} - \Delta x^a - x^{\bar{a}}$. Finally, we can calculate $u^{a'}$ in terms of $x^{\bar{a}}$ and $\Delta\tau$ by inserting r' from Eq. (A7) into our equations for the four velocity:

$$\begin{aligned}u^{t'} &= \frac{Er_0}{r_0 - 2M} - \frac{2EMr\dot{r}_0}{(r_0 - 2M)^2}\Delta\tau \\ &\quad + \frac{EM\{6L^2M^2 - 9L^2Mr_0 + r_0^2[3L^2 + 2M^2 - 2(E^2 - 1)r_0^2 - 5Mr_0]\}}{r_0^4(2M - r_0)^3}\Delta\tau^2 + \dots, \\ u^{r'} &= \dot{r}_0 + \frac{r_0(L^2 - Mr_0) - 3L^2M}{r_0^4}\Delta\tau + \frac{\dot{r}_0(12L^2M - 3L^2r_0 + 2Mr_0^2)}{2r_0^5}\Delta\tau^2 + \dots, \\ u^{\theta'} &= 0, \quad u^{\phi'} = \frac{L}{r_0^2} - \frac{2L\dot{r}_0}{r_0^3}\Delta\tau + \frac{L[3(E^2 - 1)r_0^3 - 4L^2r_0 + 7Mr_0^2 + 9L^2M]}{r_0^7}\Delta\tau^2 + \dots.\end{aligned}\tag{A8}$$

Expansions of retarded and advanced points

Taking $\Delta\tau$ to have leading order ϵ , the same leading order of our Δx terms, we can further expand it in orders of ϵ , giving

$$\Delta\tau = \tau_1\epsilon + \tau_2\epsilon^2 + \tau_3\epsilon^3 + \tau_4\epsilon^4 + \dots\tag{A9}$$

Substituting $\delta x^{a'}$ obtained from Eq. (A7) and $\Delta\tau$ from Eq. (A9) into $\sigma(x, x')$, Eq. (3.10), gives $\sigma(x, x')$ as a function of Δx^a , $x^{\bar{a}}$ and the τ_n 's:

$$\begin{aligned}\sigma(x, x') &= \frac{1}{2} \left[\frac{(2M - \bar{r})\Delta t^2}{\bar{r}} + \frac{\bar{r}(\Delta r - 2\dot{r}_0\tau_1)\Delta r}{\bar{r} - 2M} + \bar{r}^2(\Delta\theta^2 + \Delta\phi^2) - (2L\Delta\phi + \tau_1 - 2E\Delta t)\tau_1 \right] \\ &\quad + \frac{1}{2} \left\{ \frac{2}{\bar{r}} \left(\frac{EM\Delta t}{\bar{r} - 2M} - L\Delta\phi \right) \tau_1 \Delta r + \frac{M(\dot{r}_0\tau_1 - \Delta r)\Delta r^2}{(\bar{r} - 2M)^2} + \bar{r} \left[(\dot{r}_0\tau_1 + \Delta r)(\Delta\theta^2 + \Delta\phi^2) - \frac{2\dot{r}_0\tau_2\Delta r}{\bar{r} - 2M} \right] \right. \\ &\quad \left. - \frac{M(\dot{r}_0\tau_1 + \Delta r)\Delta t^2}{\bar{r}^2} - 2(L\Delta\phi - E\Delta t + \tau_1)\tau_2 \right\} + \dots\end{aligned}\tag{A10}$$

If we now specify that $x^{a'}$ coincides with the point where the world line intersects with the light cone of x , we can use the equation $\sigma(x, x') = 0$ to solve for the τ_n 's in terms of Δx^a and $x^{\bar{a}}$. This gives us

$$\begin{aligned} \tau_1 &= E\Delta t - L\Delta\phi + \frac{\bar{r}\dot{r}_0\Delta r}{2M - \bar{r}} \pm \rho, \\ \tau_2 &= \frac{\pm 1}{8\rho} \left(\frac{\{L^2 [\Delta r^2 + 4M^2 (\Delta\theta^2 + 3\Delta\phi^2)] - 4LM^2(E\Delta t \pm 2\rho)\Delta\phi - 2EM^2(3E\Delta t \pm 2\rho)\Delta t\} \Delta r}{M^2\bar{r}} \right. \\ &\quad + \frac{4LM^2\dot{r}_0\Delta t^2\Delta\phi - 4M^2[\dot{r}_0(E\Delta t \pm \rho) + 2\Delta r]\Delta t^2 + L^2\Delta r^3}{M\bar{r}^2} \\ &\quad + \frac{[4LM^2(E\Delta t - 2\dot{r}_0\Delta r)\Delta\phi + 8E^2M^3\bar{r}(\Delta\theta^2 + \Delta\phi^2) + L^2\Delta r^2 - 2EM^2(3E\Delta t \pm 2\rho)\Delta t] \Delta r}{M^2(2M - \bar{r})} \\ &\quad - \frac{\{(\bar{r} - 2M)[4LM^2\dot{r}_0\Delta\phi + 4M^2(E\dot{r}_0\Delta t \mp \dot{r}_0\rho + E^2\Delta r) - L^2\Delta r] + 8E^2M^3\Delta r\} \Delta r^2}{M(\bar{r} - 2M)^3} - \frac{4L^2M\Delta r\Delta t^2}{\bar{r}^4} \\ &\quad \left. - 4\bar{r}(\Delta\theta^2 + \Delta\phi^2)[(E^2 - 2)\Delta r - \dot{r}_0(E\Delta t - L\Delta\phi \pm \rho)] \right), \end{aligned} \quad (\text{A11})$$

with the higher order terms following in the same manner.

Using our equations for $\Delta\tau$, Eqs. (A9) and (A11), we rewrite $x^{a'}$ (and consequently $\delta x^{a'}$), the four velocity, $u^{a'}$, $\Delta^{\frac{1}{2}}(x, x')$ and $\sigma_{a'}$ (Eqs. (A7), (A8), (3.16) and (3.14) respectively), in terms of Δx^a and $x^{\bar{a}}$.

Bivector of Parallel Transport

To calculate the bivector of parallel transport, $g^{a'b'}(x, x')$, we first write it in terms of a coordinate expansion about x ,

$$g^{a'b'}(x, x') = \delta^{a'b'} + G^a{}_{bc}(x)\delta x^{c'} + G^a{}_{bcd}(x)\delta x^{c'}\delta x^{d'} + G^a{}_{bcde}(x)\delta x^{c'}\delta x^{d'}\delta x^{e'} + \dots, \quad (\text{A12})$$

where the coefficients $G^a{}_{b\dots}(x)$ are functions of x^a written in terms of Δx^a and $x^{\bar{a}}$. Calculating $g^{a'b',c'}(x, x')$ is straight forward:

$$g^{a'b',c'}(x, x') = G^a{}_{bc}(x) + 2G^a{}_{bcd}(x)\delta x^{d'} + 3G^a{}_{bcde}(x)\delta x^{d'}\delta x^{e'} + 4G^a{}_{bcde}(x)\delta x^{d'}\delta x^{e'}\delta x^{f'} + \dots. \quad (\text{A13})$$

Using the identity $g^{a'b';c'}\sigma^{c'} = g^{a'b',c'}\sigma^{c'} - \Gamma^{d'}_{b'c'}g^a_{d'}\sigma^{c'} = 0$ with Eqs. (3.12), (A13), (A12), and our expression for $\delta x^{a'}$ (obtained from the previous section), one can calculate the above coefficients and hence obtain the bivector of parallel transport, $g^{a'b'}(x, x')$, in terms of Δx^a and $x^{\bar{a}}$.

Scalar Singular Field

Combining Eqs. (2.13) and (3.1), the scalar singular field can be written as

$$\Phi^{(\text{S})}(x) = \frac{q}{2} \left[\frac{\Delta^{\frac{1}{2}}(x, x')}{\sigma_{c'}(x, x')u^{c'}(x')} \right]_{x'=x_{(\text{ret})}}^{x'=x_{(\text{adv})}} + \frac{q}{2} \int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V(x, x(\tau')) d\tau. \quad (\text{A14})$$

We already have everything required for the first term here, which gives the direct part of the scalar singular field. It should be noted that $x' = x_{(\text{ret})}$ and $x' = x_{(\text{adv})}$ are the equivalent of setting $\pm\rho = -\rho$ and $\pm\rho = +\rho$ respectively when substituting the τ_n , Eq. (A11), into $\Delta^{1/2}$, $\sigma_{a'}$ and $u^{a'}$.

In the scalar case, Eq. (3.5) for the scalar tail part becomes

$$V(x, x') = \sum_{n=0}^{\infty} V_n(x, x')\sigma^n(x, x'). \quad (\text{A15})$$

To calculate coordinate expansions of the V_n , first we require a coordinate expansion for V_0 about x of the form,

$$V_0(x, x') = v_0(x) + v_{0a}(x)\delta x^{a'} + v_{0ab}(x)\delta x^{a'}\delta x^{b'} + v_{0abc}(x)\delta x^{a'}\delta x^{b'}\delta x^{c'} + \dots . \quad (\text{A16})$$

The ‘initial condition’ described by Eq. (3.6b) in the scalar case then becomes

$$2\sigma^{;a'} V_{0;a'} - 2V_0 \Delta^{-\frac{1}{2}} \sigma^{;a'} \Delta^{\frac{1}{2};a'} + 2V_0 + (\square' - m^2 - \xi R) \Delta^{\frac{1}{2}} = 0, \quad (\text{A17})$$

and from this, it is quite simple to read off expressions for the coefficients $v_{0a\dots}$. Once we have V_0 to the desired order, we compute a coordinate expansion for V_n ($n > 0$) of the form

$$V_n(x, x') = v_n(x) + v_{na}(x)\delta x^{a'} + v_{nab}(x)\delta x^{a'}\delta x^{b'} + v_{nabc}(x)\delta x^{a'}\delta x^{b'}\delta x^{c'} + \dots . \quad (\text{A18})$$

The recursion relation for V_n , Eq. (3.6a), in the scalar case is then

$$2n\sigma^{;a'} V_{n;a'} - 2nV_n \Delta^{-\frac{1}{2}} \sigma^{;a'} \Delta^{\frac{1}{2};a'} + 2n(n+1)V_n + (\square' - m^2 - \xi R)V_{n-1} = 0, \quad (\text{A19})$$

from which we can obtain expressions for the coefficients, $v_{na\dots}$. Here, the number of terms which must be computed is determined by the accuracy to which we require the singular field. For the present calculation, we require up to v_{0abcde} , v_{1abc} and v_{2a} .

Once we have V_n to the required n , using Eqs. (3.10) and (A15), along with our expression for $\delta x^{a'}$ obtained from Eq. (A7), we get $V(x, x')$ in terms of $\Delta\tau$, Δx^a and x^a . This can be easily integrated over τ as required by Eq. (A14). Our final expression for $\Phi^{(S)}(x)$ is then obtained by using Eqs. (A9) and (A11) to remove the $\Delta\tau$ dependence. As before $\tau_{(\text{ret})}$ and $\tau_{(\text{adv})}$ are obtained by allowing $\pm\rho = -\rho$ and $\pm\rho = +\rho$, respectively.

Electromagnetic Singular Field

For the electromagnetic singular field, we use Eqs. (2.18) and (3.1) to give

$$A_a^{(\text{S})} = \frac{e}{2} \left[\frac{\Delta^{\frac{1}{2}}(x, x') g_{aa'}(x, x') u^{a'}(x')}{\sigma_{c'}(x, x') u^{c'}(x')} \right]_{x'=x_{(\text{ret})}}^{x'=x_{(\text{adv})}} + \frac{e}{2} \int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V_{aa'}(x, z(\tau)) u^{a'}(x') d\tau, \quad (\text{A20})$$

where $V_{aa'}(x, z(\tau'))$ is given by Eq. (3.5),

$$V^{aa'}(x, x') = \sum_{n=0}^{\infty} V_n^{aa'}(x, x') \sigma^n(x, x'), \quad (\text{A21})$$

and the relevant metrics at x and x' can be used to lower indices. We require a coordinate expansion of $V_0^{aa'}$ of the form,

$$V_0^{aa'}(x, x') = v_0^{aa'}(x) + v_0^{aa'}{}_{b}(x)\delta x^{b'} + v_0^{aa'}{}_{bc}(x)\delta x^{b'}\delta x^{c'} + v_0^{aa'}{}_{bcd}(x)\delta x^{b'}\delta x^{c'}\delta x^{d'} + \dots . \quad (\text{A22})$$

Substituting this into the “initial condition” in Eq. (3.6b), which in the electromagnetic case is

$$2\sigma^{;b'} V_0^{aa'}{}_{;b'} - 2V_0^{aa'} \Delta^{-\frac{1}{2}} \sigma^{;b'} \Delta^{\frac{1}{2};b'} + 2V_0^{aa'} + (\delta^{a'}{}_{b'} \square' - R^{a'}{}_{b'}) (\Delta^{\frac{1}{2}} g^{ab'}) = 0, \quad (\text{A23})$$

the coefficients of Eq. (A22), $v_0^{aa'}{}_{b\dots}$, can easily be recursively obtained. It should be noted that the covariant derivatives do require the appropriate Christoffel symbols, which can be obtained from the suitable metric at x' . Next, we construct coordinate expansions for the $V_n^{aa'}$. These have the form

$$V_n^{aa'}(x, x') = v_n^{aa'}(x) + v_n^{aa'}{}_{b}(x)\delta x^{b'} + v_n^{aa'}{}_{bc}(x)\delta x^{b'}\delta x^{c'} + v_n^{aa'}{}_{bcd}(x)\delta x^{b'}\delta x^{c'}\delta x^{d'} + \dots . \quad (\text{A24})$$

Substituting Eq. (A24) into the recursion relation (3.6a), which for the electromagnetic case becomes

$$2n\sigma^{;b'} V_n^{aa'}{}_{;b'} - 2nV_n^{aa'} \Delta^{-\frac{1}{2}} \sigma^{;b'} \Delta^{\frac{1}{2};b'} + 2n(n+1)V_n^{aa'} + (\delta^{a'}{}_{b'} \square' - R^{a'}{}_{b'}) V_{n-1}^{ab'} = 0, \quad (\text{A25})$$

we can recursively solve for the coefficients of Eq. (A24), $v_n^{aa'}{}_{b\dots}$. Once we have $V_n^{aa'}$ to the required n , we carry out the same remaining steps as in the scalar case and use Eq. (A20) to calculate the electromagnetic singular field.

Gravitational Singular Field

In the gravitational case, Eqs. (2.23) and (3.1) give

$$\bar{h}_{ab}^{(S)} = 2\mu \left[\frac{\Delta^{\frac{1}{2}}(x, x') g_{a'(a} g_{b)b'}(x, x') u^{a'}(x') u^{b'}(x')}{\sigma_{c'}(x, x') u^{c'}(x')} \right]_{x'=x_{(\text{ret})}}^{x'=x_{(\text{adv})}} + 2\mu \int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V_{aba'b'}(x, z(\tau')) u^{a'}(x') u^{b'}(x') d\tau, \quad (\text{A26})$$

where $V_{aba'b'}(x, z(\tau'))$ is given by Eq. (3.5). For the gravitational case, this is

$$V^{aba'b'}(x, x') = \sum_{n=0}^{\infty} V_n^{aba'b'}(x, x') \sigma^n(x, x'), \quad (\text{A27})$$

where the appropriate metric at x or x' can be used to lower indices. The coordinate expansion for $V_0^{aba'b'}(x, x')$ is of the form

$$V_0^{aba'b'}(x, x') = v_0^{aba'b'}(x) + v_0^{aba'b'}{}_{;c}(x) \delta x^{c'} + v_0^{aba'b'}{}_{cd}(x) \delta x^{c'} \delta x^{d'} + v_0^{aba'b'}{}_{cde}(x) \delta x^{c'} \delta x^{d'} \delta x^{e'} + \dots. \quad (\text{A28})$$

We replace $V_0^{aba'b'}(x, x')$ with Eq (A28) in the “initial condition” described by Eq. (3.6b), which for the gravitational case is

$$2\sigma^{;c'} V_0^{aba'b'}{}_{;c'} - 2V_0^{aba'b'} \Delta^{-\frac{1}{2}} \sigma^{;c'} \Delta^{\frac{1}{2}}{}_{;c'} + 2V_0^{aba'b'} + \left(\delta^{a'}{}_{c'} \delta^{b'}{}_{d'} \square' + 2C^{a'}{}_{c'}{}^{b'}{}_{d'} \right) \left(\Delta^{\frac{1}{2}} g^{c'(a} g^{b)d'} \right) = 0. \quad (\text{A29})$$

This equation may be used to recursively solve for the coefficients of Eq. (A28), $v_0^{aba'b'}{}_{c\dots}$. Next, the coordinate expansion of $V_n^{aba'b'}(x, x')$ for $n > 0$ has the form,

$$V_n^{aba'b'}(x, x') = v_n^{aba'b'}{}_{0}(x) + v_n^{aba'b'}{}_{c}(x) \delta x^{c'} + v_n^{aba'b'}{}_{cd}(x) \delta x^{c'} \delta x^{d'} + v_n^{aba'b'}{}_{cde}(x) \delta x^{c'} \delta x^{d'} \delta x^{e'} + \dots. \quad (\text{A30})$$

Substituting this into the recursion relation of Eq. (3.6a), which for the gravitational case has the form

$$2n\sigma^{;c'} V_n^{aba'b'}{}_{;c'} - 2nV_n^{aba'b'} \Delta^{-\frac{1}{2}} \sigma^{;c'} \Delta^{\frac{1}{2}}{}_{;c'} + 2n(n+1)V_n^{aba'b'} + \left(\delta^{a'}{}_{c'} \delta^{b'}{}_{d'} \square' + 2C^{a'}{}_{c'}{}^{b'}{}_{d'} \right) V_{n-1}^{abc'd'} = 0, \quad (\text{A31})$$

we can recursively solve for the coefficients of Eq. (A30), $v_n^{aba'b'}{}_{c\dots}$. As in the previous two cases, once we have $V_n^{aba'b'}(x, x')$ for the required n , it is straightforward to calculate the singular field using Eq. (A26).

Appendix B: Covariant bitensor expansions

In this Appendix, we give covariant expansions for the bitensors appearing in the formal expression for the singular field, Eq. (2.6). These are given in terms of the biscalars $\bar{s} \equiv (g^{\bar{a}\bar{b}} + u^{\bar{a}} u^{\bar{b}}) \sigma_{\bar{a}} \sigma_{\bar{b}}$, (the projection of $\sigma_{\bar{a}}(x, \bar{x})$ orthogonal to the worldline), and $\bar{r} = \sigma_{\bar{a}} u^{\bar{a}}$ (the projection of $\sigma_{\bar{a}}(x, \bar{x})$ along the worldline). In writing the coefficients, we use the notation $[T_{a_1 \dots a_n}]_{(k)}$ to denote the term of order ϵ^k in the expansion of the tensor $T_{a_1 \dots a_n}$, so that

$$T_{a_1 \dots a_n} = \sum_{k=0}^{\infty} [T_{a_1 \dots a_n}]_{(k)} \epsilon^k. \quad (\text{B1})$$

Advanced and retarded points

Eq. (2.6) for the singular field includes bitensors at points x' on the world-line between the advanced and retarded points of x . We consolidate this dependence to a single arbitrary point, \bar{x} , on the world by expanding the dependence on x' about \bar{x} . Denoting the proper distance along the world-line between $x_{(\text{adv})}/x_{(\text{ret})}$ and \bar{x} by $\Delta\tau$, we may write the expansion of this distance in powers of ϵ as

$$\Delta\tau_{(1)} = \bar{r} \pm \bar{s}, \quad \Delta\tau_{(2)} = 0, \quad \Delta\tau_{(3)} = \mp \frac{(\bar{r} \pm \bar{s})^2 R_{u\sigma u\sigma}}{6\bar{s}}, \quad \Delta\tau_{(4)} = \mp \frac{(\bar{r} \pm \bar{s})^2 ((\bar{r} \pm \bar{s}) R_{u\sigma u\sigma;u} - R_{u\sigma u\sigma;\sigma})}{24\bar{s}},$$

$$\begin{aligned}
\Delta\tau_{(5)} &= \mp \frac{(\bar{r} \pm \bar{s})^2}{360\bar{s}^3} \left\{ 5R_{u\sigma u\sigma} R_{u\sigma u\sigma} (\bar{r} \mp 3\bar{s})(\bar{r} \pm \bar{s}) + \bar{s}^2 \left[(\bar{r} \pm \bar{s})^2 \left(3R_{u\sigma u\sigma;uu} + 4R_{u\sigma u\bar{a}} R_{u\sigma u}{}^{\bar{a}} \right) \right. \right. \\
&\quad \left. \left. - (\bar{r} \pm \bar{s}) \left(3R_{u\sigma u\sigma;u\sigma} - 16R_{u\sigma u\bar{a}} R_{u\sigma\sigma}{}^{\bar{a}} \right) + 3R_{u\sigma u\sigma;\sigma\sigma} + 4R_{u\sigma\sigma\bar{a}} R_{u\sigma\sigma}{}^{\bar{a}} \right] \right\}, \\
\Delta\tau_{(6)} &= \pm \frac{(\bar{r} \pm \bar{s})^2}{4320\bar{s}^3} \left\{ 30R_{u\sigma u\sigma} \left[R_{u\sigma u\sigma;\sigma} (\bar{r} \mp 3\bar{s})(\bar{r} \pm \bar{s}) - R_{u\sigma u\sigma;u} (\bar{r} \mp 4\bar{s})(\bar{r} \pm \bar{s})^2 \right] \right. \\
&\quad + \bar{s}^2 \left[6R_{u\sigma u\sigma;\sigma\sigma\sigma} + 36R_{u\sigma\sigma\bar{a};\sigma} R_{u\sigma\sigma}{}^{\bar{a}} \right. \\
&\quad - 2(\bar{r} \pm \bar{s}) \left(3R_{u\sigma u\sigma;u\sigma\sigma} - 36R_{u\sigma\sigma\bar{a};\sigma} R_{u\sigma u}{}^{\bar{a}} - R_{u\sigma\sigma}{}^{\bar{a}} (16R_{u\sigma u\bar{a};\sigma} + 5R_{u\sigma u\sigma;\bar{a}} - 10R_{u\sigma\sigma\bar{a};u}) \right) \\
&\quad + 2(\bar{r} \pm \bar{s})^2 \left(3R_{u\sigma u\sigma;uu\sigma} - 30R_{u\sigma u\bar{a};u} R_{u\sigma\sigma}{}^{\bar{a}} + R_{u\sigma u}{}^{\bar{a}} (13R_{u\sigma u\bar{a};\sigma} + 5R_{u\sigma u\sigma;\bar{a}} - 25R_{u\sigma\sigma\bar{a};u}) \right) \\
&\quad \left. \left. - 6(\bar{r} \pm \bar{s})^3 (R_{u\sigma u\sigma;uuu} + 6R_{u\sigma u\bar{a};u} R_{u\sigma u}{}^{\bar{a}}) \right] \right\}. \tag{B2}
\end{aligned}$$

Advanced and retarded distances

Taking two derivatives of the world function, we obtain a bitensor that has the covariant expansion

$$\begin{aligned}
\sigma_{ab} &= g_{ab} - \frac{1}{3} R_{acbd} \sigma^c \sigma^d + \frac{1}{12} R_{acbd;e} \sigma^c \sigma^d \sigma^e + \left(\frac{1}{45} R_{acpd} R^p{}_{ebf} + \frac{1}{60} R_{acbd;ef} \right) \sigma^c \sigma^d \sigma^e \sigma^f \\
&\quad + \left(\frac{1}{120} R_{acpd;e} R^p{}_{fbg} + \frac{1}{120} R_{acpd} R^p{}_{ebf;g} + \frac{1}{360} R_{acbd;efg} \right) \sigma^c \sigma^d \sigma^e \sigma^f \sigma^g \\
&\quad - \left(\frac{2}{945} R_{acpd} R^p{}_{eqf} R^q{}_{gbh} + \frac{1}{504} R_{acpd;ef} R^p{}_{gbh} + \frac{17}{5040} R_{acpd;e} R^p{}_{fbg;h} \right. \\
&\quad \left. + \frac{1}{504} R_{acpd} R^p{}_{ebf;gh} + \frac{1}{2520} R_{acbd;efgh} \right) \sigma^c \sigma^d \sigma^e \sigma^f \sigma^g \sigma^h \\
&\quad + \left(\frac{17}{20160} R_{acpd;e} R^p{}_{fqq} R^q{}_{hbi} + \frac{29}{30240} R_{acpd} R^p{}_{eqf;g} R^q{}_{hbi} + \frac{11}{30240} R_{acpd;efg} R^p{}_{hbi} \right. \\
&\quad \left. + \frac{17}{20160} R_{acpd;ef} R^p{}_{gbh;i} + \frac{17}{20160} R_{acpd} R^p{}_{eqf} R^q{}_{gbh;i} + \frac{17}{20160} R_{acpd;e} R^p{}_{fbg;hi} \right. \\
&\quad \left. + \frac{11}{30240} R_{acpd} R^p{}_{ebf;ghi} + \frac{1}{20160} R_{acbd;efghi} \right) \sigma^c \sigma^d \sigma^e \sigma^f \sigma^g \sigma^h \sigma^i. \tag{B3}
\end{aligned}$$

For the singular field, we require the expansion of $[\sigma_{a'} u^{a'}](z_{\pm}, x)$. Writing $[\sigma_{a'} u^{a'}](\tau) = [\sigma_{a'} u^{a'}](z(\tau), x)$, expanding the dependence on τ about \bar{x} (using the method of Sec. III A and making use of the above expansion of σ_{ab}) and evaluating at $\tau = \tau_{\pm}$, we obtain the coefficients of the expansion of $[\sigma_{a'} u^{a'}]_{\pm} \equiv [\sigma_{a'} u^{a'}](z_{\pm}, x)$ about \bar{x} . They are:

$$\begin{aligned}
r_{(1)} &= \bar{s}, \quad r_{(3)} = -\frac{\bar{r}^2 - \bar{s}^2}{6\bar{s}} R_{u\sigma u\sigma}, \quad r_{(4)} = \frac{\bar{r} \pm \bar{s}}{24\bar{s}} \left[R_{u\sigma u\sigma\sigma} (\bar{r} \mp 2\bar{s}) - R_{u\sigma u\sigma u} (\bar{r} \pm \bar{s})(\bar{r} \mp 2\bar{s}) \right], \\
r_{(5)} &= -\frac{1}{360\bar{s}^3} \left\{ \bar{s}^2 \left[(\bar{r}^2 - \bar{s}^2)(3R_{u\sigma u\sigma\sigma\sigma} + 4R_{u\sigma\sigma}{}^{\bar{a}} R_{u\sigma u\bar{a}}) - (\bar{r} \pm \bar{s})^2 (\bar{r} \mp 2\bar{s})(3R_{u\sigma u\sigma\sigma u} - 16R_{u\sigma\sigma}{}^{\bar{a}} R_{u\sigma u\bar{a}}) \right. \right. \\
&\quad \left. \left. + (\bar{r} \pm \bar{s})^3 (\bar{r} \mp 3\bar{s})(3R_{u\sigma u\sigma u u} + 4R_{u\sigma u}{}^{\bar{a}} R_{u\sigma u\bar{a}}) \right] + 5 \left[(\bar{r}^2 - \bar{s}^2) R_{u\sigma u\sigma} \right]^2 \right\}, \\
r_{(6)} &= \frac{1}{4320\bar{s}^3} \left\{ \bar{s}^2 \left[6(\bar{r}^2 - \bar{s}^2)(R_{u\sigma u\sigma\sigma\sigma\sigma} + 6R_{u\sigma\sigma}{}^{\bar{a}} R_{u\sigma u\bar{a}}) - 6(\bar{r} \pm \bar{s})^4 (\bar{r} \mp 4\bar{s})(R_{u\sigma u\sigma u u u} + 6R_{u\sigma u\bar{a} u} R_{u\sigma u}{}^{\bar{a}}) \right. \right. \\
&\quad \left. \left. - 2(\bar{r} \pm \bar{s})^2 (\bar{r} \mp 2\bar{s})(3R_{u\sigma u\sigma u\sigma\sigma} - 36R_{u\sigma\sigma\bar{a}\sigma} R_{u\sigma u}{}^{\bar{a}} - R_{u\sigma\sigma}{}^{\bar{a}} (16R_{u\sigma u\bar{a}\sigma} + 5R_{u\sigma u\sigma\bar{a}} - 10R_{u\sigma\sigma\bar{a}\bar{a}})) \right. \right. \\
&\quad \left. \left. + 2(\bar{r} \pm \bar{s})^3 (\bar{r} \mp 3\bar{s})(3R_{u\sigma u\sigma u u\sigma} - 30R_{u\sigma u\bar{a} u} R_{u\sigma\sigma}{}^{\bar{a}} + R_{u\sigma u}{}^{\bar{a}} (13R_{u\sigma u\bar{a}\sigma} + 5R_{u\sigma u\sigma\bar{a}} - 25R_{u\sigma\sigma\bar{a}\bar{a}})) \right] \right. \\
&\quad \left. + 30R_{u\sigma u\sigma} \left[(\bar{r}^2 - \bar{s}^2)^2 R_{u\sigma u\sigma\sigma} - (\bar{r} \pm \bar{s})^3 (\bar{r}^2 \mp 3\bar{r}\bar{s} + 4\bar{s}^2) R_{u\sigma u\sigma u} \right] \right\}. \tag{B4}
\end{aligned}$$

Van Vleck Determinant

The Van Vleck determinant has the covariant expansion

$$\Delta^{1/2}(x, x') = 1 + \frac{1}{12} R_{ab} \sigma^a \sigma^b - \frac{1}{24} R_{ab;c} \sigma^a \sigma^b \sigma^c + \left(\frac{1}{360} R_{paqb} R^p{}_c{}^q{}_d + \frac{1}{288} R_{ab} R_{cd} + \frac{1}{80} R_{ab;cd} \right) \sigma^a \sigma^b \sigma^c \sigma^d$$

$$\begin{aligned}
& - \left(\frac{1}{360} R_{paqb} R^p{}_c{}^q{}_{d;e} + \frac{1}{288} R_{ab} R_{cd;e} + \frac{1}{360} R_{ab;cd e} \right) \sigma^a \sigma^b \sigma^c \sigma^d \sigma^e \\
& + \left(\frac{1}{1260} R_{paqb} R^p{}_c{}^q{}_{d;ef} + \frac{1}{1344} R_{paqb;c} R^p{}_d{}^q{}_{e;f} + \frac{1}{5670} R_{paqb} R_{rc}{}^p{}_d R^q{}_e{}^r{}_f + \frac{1}{4320} R_{paqb} R^p{}_c{}^q{}_d R_{ef} \right. \\
& \quad \left. + \frac{1}{10368} R_{ab} R_{cd} R_{ef} + \frac{1}{1152} R_{ab;c} R_{de;f} + \frac{1}{960} R_{ab} R_{cd;ef} + \frac{1}{2016} R_{ab;cd e f} \right) \sigma^a \sigma^b \sigma^c \sigma^d \sigma^e \sigma^f \\
& - \left(\frac{1}{6048} R_{paqb} R^p{}_c{}^q{}_{d;ef g} + \frac{1}{2240} R_{paqb;c} R^p{}_d{}^q{}_{e;fg} + \frac{1}{3780} R_{paqb} R_{rc}{}^p{}_d R^q{}_e{}^r{}_f{}_g \right. \\
& \quad \left. + \frac{1}{4320} R_{paqb} R^p{}_c{}^q{}_{d;e} R_{fg} + \frac{1}{8640} R_{paqb} R^p{}_c{}^q{}_d R_{ef;g} + \frac{1}{6912} R_{ab} R_{cd} R_{ef;g} \right. \\
& \quad \left. + \frac{1}{1920} R_{ab;c} R_{de;fg} + \frac{1}{4320} R_{ab} R_{cd;ef g} + \frac{1}{13440} R_{ab;cd e f g} \right) \sigma^a \sigma^b \sigma^c \sigma^d \sigma^e \sigma^f \sigma^g. \tag{B5}
\end{aligned}$$

Writing $\Delta^{1/2}(\tau) = \Delta^{1/2}(z(\tau), x)$, expanding the dependence on τ about \bar{x} (using the method of Sec. III A) and evaluating at $\tau = \tau_{\pm}$, we obtain the coefficients in the expansion of $\Delta_{\pm}^{1/2} \equiv \Delta^{1/2}(z_{\pm}, x)$ about \bar{x} . Specialized to the vacuum case, they are:

$$\Delta_{(0)}^{1/2} = 1, \quad \Delta_{(1)}^{1/2} = 0, \quad \Delta_{(2)}^{1/2} = 0, \quad \Delta_{(3)}^{1/2} = 0,$$

$$\begin{aligned}
\Delta_{(4)}^{1/2} &= \frac{1}{360} \left[C_{\sigma \bar{a} \sigma \bar{b}} + 2(\bar{r} \pm \bar{s}) C_{u(\bar{a}|\sigma|\bar{b})} + (\bar{r} \pm \bar{s})^2 C_{u \bar{a} u \bar{b}} \right] \left[C_{\sigma}{}^{\bar{a}}{}_{\sigma}{}^{\bar{b}} + 2(\bar{r} \pm \bar{s}) C_u{}^{\bar{a}}{}_{\sigma}{}^{\bar{b}} + (\bar{r} \pm \bar{s})^2 C_u{}^{\bar{a}}{}_{u}{}^{\bar{b}} \right], \\
\Delta_{(5)}^{1/2} &= \frac{1}{360} \left[C_{\sigma \bar{a} \sigma \bar{b}} + 2(\bar{r} \pm \bar{s}) C_{u(\bar{a}|\sigma|\bar{b})} + (\bar{r} \pm \bar{s})^2 C_{u \bar{a} u \bar{b}} \right] \left[(\bar{r} \pm \bar{s}) \left(C_{\sigma}{}^{\bar{a}}{}_{\sigma}{}^{\bar{b}} u - 2 C_u{}^{\bar{a}}{}_{\sigma}{}^{\bar{b}} \sigma \right) \right. \\
&\quad \left. - (\bar{r} \pm \bar{s})^2 \left(C_u{}^{\bar{a}}{}_{u}{}^{\bar{b}} \sigma - 2 C_u{}^{\bar{a}}{}_{\sigma}{}^{\bar{b}} u \right) + (\bar{r} \pm \bar{s})^3 C_u{}^{\bar{a}}{}_{u}{}^{\bar{b}} u - C_{\sigma}{}^{\bar{a}}{}_{\sigma}{}^{\bar{b}} \sigma \right]. \tag{B6}
\end{aligned}$$

Bivector of parallel transport

The derivative of the bivector of parallel transport has the covariant expansion

$$\begin{aligned}
g_a{}^{a'} g_{a'b;c}(x, x') &= -\frac{1}{2} R_{bacd} \sigma^d + \frac{1}{6} R_{bacd;e} \sigma^d \sigma^e - \frac{1}{24} \left(R_{bapd} R^p{}_{ecf} + R_{bacd;ef} \right) \sigma^d \sigma^e \sigma^f \\
&\quad + \left(\frac{1}{60} R_{bapd} R^p{}_{ecf;g} + \frac{7}{360} R_{bapd;e} R^p{}_{fcg} + \frac{1}{120} R_{bacd;efg} \right) \sigma^d \sigma^e \sigma^f \sigma^g \\
&\quad - \left(\frac{1}{240} R_{bapd} R^p{}_{ecf;gh} - \frac{1}{120} R_{bapd;e} R^p{}_{fcg;h} + \frac{1}{180} R_{bapd;ef} R^p{}_{gch} \right. \\
&\quad \left. + \frac{1}{240} R_{bapd} R^p{}_{eqf} R^q{}_{gch} + \frac{1}{720} R_{bacd;efgh} \right) \sigma^d \sigma^e \sigma^f \sigma^g \sigma^h. \tag{B7}
\end{aligned}$$

For the singular field, we require the expansion of $g_{aa'} u^{a'}(z_{\pm}, x)$. Writing $[g_{aa'} u^{a'}](\tau) = [g_{aa'} u^{a'}](z(\tau), x)$, expanding the dependence on τ about \bar{x} (using the method of Sec. III A and making use of the above expansion of the bivector of parallel transport) and evaluating at $\tau = \tau_{\pm}$, we obtain the coefficients of the expansion of $[g_{aa'} u^{a'}]_{\pm} \equiv g_{aa'} u^{a'}(z_{\pm}, x)$ about \bar{x} . They are:

$$\begin{aligned}
\left[g_{aa'} u^{a'} \right]_{(0)} &= g_{a \bar{a}} u^{\bar{a}}, \quad \left[g_{aa'} u^{a'} \right]_{(1)} = 0, \quad \left[g_{aa'} u^{a'} \right]_{(2)} = -\frac{1}{2} (\bar{r} \pm \bar{s}) g_a{}^{\bar{a}} R_{u \sigma u \bar{a}}, \\
\left[g_{aa'} u^{a'} \right]_{(3)} &= \frac{1}{6} (\bar{r} \pm \bar{s}) g_a{}^{\bar{a}} (R_{u \sigma u \bar{a};\sigma} - (\bar{r} \pm \bar{s}) R_{u \sigma u \bar{a};u}),
\end{aligned}$$

$$\begin{aligned}
\left[g_{aa'} u^{a'} \right]_{(4)} &= \pm \frac{1}{24\bar{s}} g_a{}^{\bar{a}} (\bar{r} \pm \bar{s}) \left\{ 2(\bar{r} \pm \bar{s}) R_{u \sigma u \bar{a}} R_{u \sigma u \sigma} - \bar{s} \left[R_{u \sigma u \bar{a};\sigma \sigma} + R_{u \bar{a} \sigma \bar{b}} R_{u \sigma \sigma}{}^{\bar{b}} + (\bar{r} \pm \bar{s})^2 R_{u \sigma u a;uu} \right. \right. \\
&\quad \left. \left. + (\bar{r} \pm \bar{s}) \left(R_{u \sigma u}{}^{\bar{b}} (2 R_{u \bar{a} \sigma \bar{b}} + 3 R_{u \sigma \bar{a} \bar{b}}) - R_{u \sigma u \bar{a};u \sigma} + R_{u \bar{a} u}{}^{\bar{b}} (R_{u \sigma \sigma \bar{b}} + (\bar{r} \pm \bar{s}) R_{u \sigma u \bar{b}}) \right) \right] \right\}, \\
\left[g_{aa'} u^{a'} \right]_{(5)} &= \pm \frac{1}{2160\bar{s}} g_a{}^{\bar{a}} (\bar{r} \pm \bar{s}) \left\{ \bar{s} \left[18 \left(R_{u \sigma u a;\sigma \sigma \sigma} - R_{u \sigma u a;u \sigma \sigma} (\bar{r} \pm \bar{s}) + R_{u \sigma u a;uu \sigma} (\bar{r} \pm \bar{s})^2 - R_{u \sigma u a;uuu} (\bar{r} \pm \bar{s})^3 \right) \right. \right. \\
&\quad \left. \left. + (\bar{r} \pm \bar{s}) \left(R_{u \sigma u}{}^{\bar{b}} (2 R_{u \bar{a} \sigma \bar{b}} + 3 R_{u \sigma \bar{a} \bar{b}}) - R_{u \sigma u \bar{a};u \sigma} + R_{u \bar{a} u}{}^{\bar{b}} (R_{u \sigma \sigma \bar{b}} + (\bar{r} \pm \bar{s}) R_{u \sigma u \bar{b}}) \right) \right] \right\},
\end{aligned}$$

$$\begin{aligned}
& +6R_{u\sigma\sigma b}\left(7R_{ua\sigma}{}^b{}_{;\sigma}+(\bar{r}\pm\bar{s})(3R_{uau}{}^b{}_{;\sigma}-4R_{ua\sigma}{}^b{}_{;u}+4R_{u\sigma ua}{}^{;b})-8R_{uau}{}^b{}_{;u}(\bar{r}\pm\bar{s})^2\right) \\
& +9R_{ua\sigma b}\left(4R_{u\sigma\sigma}{}^b{}_{;\sigma}+(\bar{r}\pm\bar{s})(5R_{u\sigma u}{}^b{}_{;\sigma}-3R_{u\sigma\sigma}{}^b{}_{;u})-6R_{u\sigma u}{}^b{}_{;u}(\bar{r}\pm\bar{s})^2\right) \\
& +2R_{u\sigma ub}\left(21(\bar{r}\pm\bar{s})(2R_{ua\sigma}{}^b{}_{;\sigma}+3R_{u\sigma a}{}^b{}_{;\sigma})\right. \\
& \quad \left. +(\bar{r}\pm\bar{s})^2(17R_{uau}{}^b{}_{;\sigma}-32R_{ua\sigma}{}^b{}_{;u}-36R_{u\sigma a}{}^b{}_{;u}+16R_{u\sigma ua}{}^{;b})-21R_{uau}{}^b{}_{;u}(\bar{r}\pm\bar{s})^3\right) \\
& +9R_{uau b}\left(4R_{u\sigma\sigma}{}^b{}_{;\sigma}(\bar{r}\pm\bar{s})+3(\bar{r}\pm\bar{s})^2(R_{u\sigma u}{}^b{}_{;\sigma}-R_{u\sigma\sigma}{}^b{}_{;u})-4R_{u\sigma u}{}^b{}_{;u}(\bar{r}\pm\bar{s})^3\right) \\
& \quad +54(\bar{r}\pm\bar{s})R_{u\sigma ab}\left(R_{u\sigma u}{}^b{}_{;\sigma}-2R_{u\sigma u}{}^b{}_{;u}(\bar{r}\pm\bar{s})\right)\Big] \\
& +15(\bar{r}\pm\bar{s})\left[4R_{u\sigma u\sigma}\left(2(\bar{r}\pm\bar{s})R_{u\sigma ua;u}-R_{u\sigma ua;\sigma}\right)+3R_{u\sigma ua}\left(R_{u\sigma u\sigma;u}(\bar{r}\pm\bar{s})-R_{u\sigma u\sigma;\sigma}\right)\right]\Big\}. \quad (\text{B8})
\end{aligned}$$

Scalar tail

The scalar tail bitensor, $V(x, x')$, may be expanded in a covariant series by writing it in the form of a Hadamard series,

$$V(x, x') = V_0(x, x') + V_1(x, x')\sigma(x, x') + \dots, \quad (\text{B9})$$

and expanding each of the Hadamard coefficients $V_0(x, x'), V_1(x, x'), \dots$ in a covariant Taylor series,

$$\begin{aligned}
V_0 &= v_0 - \frac{1}{2}v_{0;c}\sigma^c + \frac{1}{2}v_{0;cd}\sigma^c\sigma^d + \frac{1}{6}\left(-\frac{3}{2}v_{0;(cd;e)} + \frac{1}{4}v_{0;(cde)}\right)\sigma^c\sigma^d\sigma^e + \dots, \\
V_1 &= v_1 - \frac{1}{2}v_{1;c}\sigma^c + \dots.
\end{aligned}$$

The series coefficients required to obtain the expansion of the singular field to $\mathcal{O}(\epsilon^4)$ [$V(x, x')$ to $\mathcal{O}(\epsilon^3)$] are given by

$$\begin{aligned}
v_0 &= \frac{1}{2}\left((\xi - \frac{1}{6})R + m^2\right), \\
v_0{}^{cd} &= -\frac{1}{180}R_{pqr}{}^c R^{pqr} - \frac{1}{180}R_p{}^c{}_p{}^d R^{pq} + \frac{1}{90}R_p{}^c{}_p R^{dp} - \frac{1}{120}\square R^{cd} + \frac{1}{12}(\xi - \frac{1}{6})RR^{cd} + \left(\frac{1}{6}\xi - \frac{1}{40}\right)R^{cd} + \frac{1}{12}m^2 R^{cd}, \\
v_1 &= \frac{1}{720}R_{pqrs}R^{pqrs} - \frac{1}{720}R_{pq}R^{pq} + \frac{1}{8}(\xi - \frac{1}{6})^2 R^2 - \frac{1}{24}(\xi - \frac{1}{5})\square R + \frac{1}{4}m^2(\xi - \frac{1}{6})R + \frac{1}{8}m^4.
\end{aligned}$$

For the singular field, we require the expansion of $\int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V d\tau'$. Writing $V(\tau) = V(z(\tau), x)$ and expanding the dependence on τ about \bar{x} (using the method of Sec. III A and making use of the above expansion of V), we obtain an expansion in powers of $\Delta\tau$ that can be trivially integrated between $\tau = \tau_-$ and $\tau = \tau_+$. Specialized to the vacuum case, the required expansion coefficients are then:

$$\left[\int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V d\tau'\right]_{(1)} = 0, \quad \left[\int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V d\tau'\right]_{(2)} = 0, \quad \left[\int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V d\tau'\right]_{(3)} = 0, \quad \left[\int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V d\tau'\right]_{(4)} = 0.$$

Electromagnetic tail

The electromagnetic tail bitensor, $V_{ab'}(x, x')$, may be expanded in a covariant series by writing it in the form of a Hadamard series,

$$V_{ab'}(x, x') = g_{b'}{}^b[V_{0ab}(x, x') + V_{1ab}(x, x')\sigma(x, x') + \dots], \quad (\text{B10})$$

and expanding each of the Hadamard coefficients, $V_{0ab}(x, x'), V_{1ab}(x, x'), \dots$, in a covariant Taylor series,

$$\begin{aligned}
V_{0;ab} &= v_{0;(ab)} + \left(-\frac{1}{2}v_{0;(ab);c} + v_{0;[ab]c}\right)\sigma^c + \frac{1}{2}\left(v_{0;(ab)cd} - v_{0;[ab](c;d)}\right)\sigma^c\sigma^d \\
& \quad + \frac{1}{6}\left(-\frac{3}{2}v_{0;(ab)(cd;e)} + \frac{1}{4}v_{0;(ab)(cde)} + v_{0;[ab]cde}\right)\sigma^c\sigma^d\sigma^e + \dots, \\
V_{1;ab} &= v_{1;(ab)} + \left(-\frac{1}{2}v_{1;(ab);c} + v_{1;[ab]c}\right)\sigma^c + \dots.
\end{aligned}$$

The series coefficients required to obtain the expansion of the singular field to $\mathcal{O}(\epsilon^4)$ [$V_{ab'}(x, x')$ to $\mathcal{O}(\epsilon^3)$] are given by

$$\begin{aligned} v_0(ab) &= \frac{1}{2}R_{ab} - \frac{1}{12}Rg_{ab}, \\ v_0[ab]^c &= \frac{1}{6}R^c_{[b;a]}, \\ v_0(ab)^{cd} &= \frac{1}{6}R_{ab}^{;(cd)} + \frac{1}{12}R_{ab}R^{cd} + \frac{1}{12}R_{(a}^{pq}R_{b)pq}{}^d \\ &\quad + g_{ab}\left(-\frac{1}{180}R_{pqr}{}^cR^{pqrd} - \frac{1}{180}R_p{}^d{}_qR^{pq} + \frac{1}{90}R_p{}^cR^{dp} - \frac{1}{72}RR^{cd} - \frac{1}{40}R^{cd} - \frac{1}{120}\square R^{cd}\right), \\ v_0[ab]^{cde} &= -\frac{3}{20}R^{(c}_{[a;b]}{}^{de)} - \frac{1}{12}R^{(c}_{[a;b]}R^{de)} - \frac{1}{20}R_{[a}^{pq(c}R_{b]pq}{}^{d;e)} - \frac{1}{30}R_{ab}{}^{p(c};{}_qR_p{}^{de)q} + \frac{1}{60}R_{abp}{}^{(c;d}R^e{}^{)p} \\ &\quad + \frac{1}{20}R_{abp}{}^{(c}R^{de);p} - \frac{1}{20}R_{ab}{}^{p(c}R_p{}^{d;e)} \end{aligned}$$

and

$$\begin{aligned} v_1(ab) &= -\frac{1}{48}R_{apqr}R_b{}^{pqr} + \frac{1}{8}R_{ap}R_b{}^p - \frac{1}{24}RR_{ab} - \frac{1}{24}\square R_{ab} \\ &\quad + g_{ab}\left(\frac{1}{720}R_{pqrs}R^{pqrs} - \frac{1}{720}R_{pq}R^{pq} + \frac{1}{288}R^2 + \frac{1}{120}\square R\right), \\ v_1[ab]^c &= \frac{1}{240}R_{[a}{}^{pqr}R_{b]pqr}{}^{;c} + \frac{1}{24}R_{[a}{}^{pq}R_{b]p;q} + \frac{1}{120}R_{[a}{}^{pq}R_{b]q;p} \\ &\quad - \frac{1}{120}R^c{}_{pq[a}R^{pq}{}_{;b]} + \frac{1}{24}R^{pc}{}_{;[a}R_{b]p} + \frac{1}{24}R^p{}_{[a}R_b{}^c{}_{;p} - \frac{1}{360}R^{pc}R_{p[a;b]} - \frac{1}{24}R^p{}_{[a}R_{b]p}{}^{;c} + \frac{1}{72}RR^c{}_{[a;b]} \\ &\quad + \frac{1}{120}R^c{}_{[a;b]}{}^p - \frac{1}{540}R_{abpq;r}R^{pqrc} - \frac{1}{540}R_{abpq;r}R^{rqpc} - \frac{1}{360}R_{abp}{}^c{}_{;q}R^{pq} + \frac{1}{120}R_{abpq}R^{cp;q} - \frac{1}{120}R_{ab}{}^{pc}R_{;p}. \end{aligned}$$

These are the same as those given by Brown and Ottewill [87] with the exception of $v_1[ab]^c$, where we have corrected a sign error in one of their terms and combined another two terms into a single term.

For the singular field, we require the expansion of $\int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V_{ab'}u^{b'}d\tau'$. Writing $[V_{ab'}u^{b'}](\tau) = [V_{ab'}u^{b'}](z(\tau), x)$ and expanding the dependence on τ about \bar{x} (using the method of Sec. III A and making use of the above expansion of $V_{ab'}$), we obtain an expansion in powers of $\Delta\tau$ that can be trivially integrated between $\tau = \tau_-$ and $\tau = \tau_+$. Specialized to the vacuum case, the required expansion coefficients are then:

$$\begin{aligned} \left[\int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V_{ab'}u^{b'}d\tau'\right]_{(1)} &= 0, \quad \left[\int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V_{ab'}u^{b'}d\tau'\right]_{(2)} = 0, \\ \left[\int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V_{ab'}u^{b'}d\tau'\right]_{(3)} &= \frac{1}{1152}(\bar{r} \pm \bar{s})g_a{}^{\bar{a}} \left[16R_{ubac}R_u{}^b{}_u{}^c(\bar{r} \pm \bar{s})^2 + 48R_u{}^b{}_\sigma{}^cR_{\sigma cab} \right. \\ &\quad \left. + 24(\bar{r} \pm \bar{s}) \left(R_u{}^b{}_\sigma{}^cR_{ucab} + R_u{}^b{}_u{}^cR_{\sigma bac} + (\bar{r} \mp 2\bar{s})R_{bcde}R^{bcde}u_a \right) \right], \\ \left[\int_{\tau_{(\text{ret})}}^{\tau_{(\text{adv})}} V_{ab'}u^{b'}d\tau'\right]_{(4)} &= \frac{1}{51840}(\bar{r} \pm \bar{s})g_a{}^{\bar{a}} \left\{ 144 \left[2R_{ua\sigma}{}^{bc}R_{\sigma b\sigma c} - 6R_\sigma{}^b{}_a{}^c{}_\sigma R_{uc\sigma b} - 9R_u{}^b{}_\sigma{}^c{}_\sigma R_{\sigma cab} + R_\sigma{}^{bcd}{}_\sigma R_{\sigma bcd}u_a \right] \right. \\ &\quad - 4(\bar{r} \pm \bar{s}) \left[18 \left(6R_\sigma{}^b{}_a{}^c{}_\sigma R_{ubuc} - 2R_{ua\sigma}{}^{bc}R_{ub\sigma c} + 9R_u{}^b{}_\sigma{}^c{}_\sigma R_{ucab} + R_{uc\sigma b}(6R_u{}^b{}_a{}^c{}_\sigma \right. \right. \\ &\quad \left. \left. - 2R_{ua\sigma}{}^{bc} - 9R_\sigma{}^b{}_a{}^c u) + 9R_u{}^b{}_u{}^c{}_\sigma R_{\sigma bac} - 2R_{ua}{}^{bc}R_{\sigma b\sigma c} - 6R_u{}^b{}_\sigma{}^c{}_u R_{\sigma cab} \right) \right. \\ &\quad + (\bar{r} \mp 2\bar{s}) \left(27R_u{}^{bcd}{}_\sigma R_{abcd} + 18R_a{}^{bcd}{}_\sigma R_{ubcd} + 4R_{ua}{}^{bcd}(R_{\sigma bcd} - R_{\sigma dbc}) \right) \\ &\quad - 3 \left(6R_\sigma{}^{bcd}{}_\sigma R_{ubcd} + 6(R_u{}^{bcd}{}_\sigma - R_\sigma{}^{bcd} u)R_{\sigma bcd} + R^{bcde}{}_\sigma R_{bcde}(\bar{r} \mp 2\bar{s}) \right) u_a \Big] \\ &\quad + (\bar{r} \pm \bar{s})^2 \left[48 \left(R_{ubuc}(2R_{ua\sigma}{}^{bc} - 6R_u{}^b{}_a{}^c{}_\sigma + 9R_\sigma{}^b{}_a{}^c u) + 6R_u{}^b{}_\sigma{}^c{}_u R_{ucab} + 9R_u{}^b{}_a{}^c{}_u R_{uc\sigma b} \right. \right. \\ &\quad \left. \left. + 2R_{ua}{}^{bc}(R_{ub\sigma c} + R_{uc\sigma b}) + 6R_u{}^b{}_u{}^c{}_\sigma R_{\sigma bac} \right) \right. \\ &\quad + (\bar{r} \mp 3\bar{s}) \left(R_{ubcd}(27R_a{}^{bcd}{}_u - 4R_{ua}{}^{bcd}) + 4R_{ua}{}^{bcd}R_{ubdc} \right) \\ &\quad + 3 \left(16(R_u{}^{bcd}{}_\sigma - R_\sigma{}^{bcd} u)R_{ubcd} - R^{bcde}{}_\sigma R_{bcde}(\bar{r} \mp 3\bar{s}) \right) u_a \\ &\quad + 6R_u{}^{bcd}{}_u \left(3R_{abcd}(\bar{r} \mp 3\bar{s}) - 8R_{\sigma bcd}u_a \right) - 432R_u{}^b{}_u{}^c{}_\sigma R_{ubac} \Big] \\ &\quad \left. + 36(\bar{r} \pm \bar{s})^3 \left[6R_u{}^b{}_u{}^c{}_\sigma R_{ubac} + R_{ubuc}(2R_{ua}{}^{bc} + 9R_u{}^b{}_a{}^c u) - R_u{}^{bcd}{}_u R_{ubcd}u_a \right] \right\}. \end{aligned} \tag{B11}$$

Gravitational tail

The gravitational tail bitensor, $V_{aa'bb'}(x, x')$, may be expanded in a covariant series by writing it in the form of a Hadamard series,

$$V_{aa'bb'}(x, x') = g_{a'}{}^c g_{b'}{}^d [V_{0acbd}(x, x') + V_{1acbd}(x, x')\sigma(x, x') + \dots], \quad (\text{B12})$$

and expanding each of the Hadamard coefficients, $V_{0acbd}(x, x'), V_{1acbd}(x, x'), \dots$, in a covariant Taylor series,

$$\begin{aligned} V_{0AB} &= v_{0(AB)} + \left(-\frac{1}{2}v_{0(AB);e} + v_{0[AB]e}\right)\sigma^e + \frac{1}{2}\left(v_{0(AB)ef} - v_{0[AB](e;f)}\right)\sigma^e\sigma^f \\ &\quad + \frac{1}{6}\left(-\frac{3}{2}v_{0(AB)(ef;g)} + \frac{1}{4}v_{0(AB);(efg)} + v_{0[AB]efg}\right)\sigma^e\sigma^f\sigma^g + \dots, \\ V_{1AB} &= v_{1(AB)} + \left(-\frac{1}{2}v_{1(AB);c} + v_{1[AB]e}\right)\sigma^e + \dots. \end{aligned}$$

The required coefficients for V_{0AB} are

$$v_{0(AB)} = v_{0(\overline{ab}\overline{cd})} = -C_{acbd}, \quad (\text{B13})$$

$$v_{0[AB]}{}^e = 0, \quad (\text{B14})$$

$$v_{0(AB)}{}^{ef} = v_{0(\overline{ab}\overline{cd})}{}^{ef} = -\frac{1}{3}C_{acbd}{}^{;(ef)} - \frac{1}{6}C_{ac}{}^{p(e}C_{bd}{}^{p)f} + \frac{1}{6}g_{ac}C_b{}^{pq(e}C_{dpq}{}^{f)} - \frac{1}{720}\Pi_{abcd}g^{ef}C^{pqrs}C_{pqrs}, \quad (\text{B15})$$

and

$$v_{0[AB]}{}^{efg} = v_{0[\overline{ab}\overline{cd}]}{}^{efg} = \frac{1}{10}g_{ac}C_b{}^{pq(e;f}C_d{}^{pq)g} + \frac{1}{15}g_{ac}C_{bdpe;q}C^p{}_f{}^q{}_g, \quad (\text{B16})$$

where

$$\Pi_{abcd} = \frac{1}{2}g_{ac}g_{bd} + \frac{1}{2}g_{ad}g_{bc} + \kappa g_{ab}g_{cd}. \quad (\text{B17})$$

In Eqs. (B13) – (B16), the right hand sides are understood to be symmetrized on the index pairs (ab) and (cd) . The required coefficients for V_{1AB} are

$$v_{1(AB)} = v_{1(\overline{ab}\overline{cd})} = \frac{1}{12}\square C_{acbd} + \frac{1}{2}C_a{}^p{}_b{}^q C_{cpdq} + \frac{1}{24}C_{ac}{}^{pq}C_{bdpq} - \frac{1}{96}g_{ac}g_{bd}C^{pqrs}C_{pqrs} + \frac{1}{720}\Pi_{abcd}C^{pqrs}C_{pqrs} \quad (\text{B18})$$

and

$$\begin{aligned} v_{1[AB]}{}^e &= v_{1[\overline{ab}\overline{cd}]}{}^e \\ &= \frac{1}{12}(C_a{}^p{}_b{}^q C_{cpdq}{}^{;e} - C_c{}^p{}_d{}^q C_{apbq}{}^{;e}) + \frac{1}{12}(C_a{}^{peq}C_{bcdp;q} - C_c{}^{peq}C_{dabp;q}) + \frac{1}{90}g_{ac}C^{epqr}C_{bdpq;r}, \end{aligned} \quad (\text{B19})$$

where again there is implicit symmetrization on the index pairs (ab) and (cd) . When $\kappa = -1/2$, Eq. (B18) agrees with Eq. (A23) of Allen, Folacci and Ottewill [88] specialized to the vacuum case. Our expressions also agree with Anderson, Flanagan and Ottewill [89], but we write them here in a slightly more compact form. Note that the expressions (B13) – (B16) and (B18) – (B19) are all traceless on the index pair (cd) , aside from the terms involving the tensor Π_{abcd} . This means that performing a trace reversal on the index pair (cd) is equivalent to changing the value of κ from 0 to $-1/2$. For the calculation of the gravitational singular field, we require an expansion of the trace reversed singular field and so we choose $\kappa = 0$.

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